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"STUDIES ON SOME COMMON FIXED-POINT THEOREMS IN UNIFORM SPACES AND THEIR USES"

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ABSTRACT:

Fixed-point theorems constitute foundational results in mathematical analysis, asserting the existence of points invariant under certain mappings. This paper investigates common fixed-point theorems within the framework of uniform spaces, emphasizing their theoretical underpinnings and practical applications. Beginning with an overview of uniform spaces and their properties, we explore prominent fixed-point theorems such as Banach's and Kakutani's, analyzing their conditions and implications. Applications of these theorems in functional analysis, optimization, and beyond are discussed, illustrating their role in addressing stability and convergence in iterative processes. By bridging theoretical insights with concrete examples, this study underscores the significance of fixed-point theorems in advancing mathematical theory and facilitating solutions to diverse real-world problems.

KEYWORDS: *Fixed-point theorems, uniform spaces and Mathematical analysis.*

INTRODUCTION

Fixed-point theorems are fundamental results in mathematics that establish the existence of points unchanged by a mapping. In the context of uniform spaces, which generalize the notion of metric spaces by defining a uniform structure that captures uniform continuity, these theorems play a crucial role in various areas of mathematical analysis and its applications.

The study of fixed-point theorems in uniform spaces encompasses a rich tapestry of results and applications. These theorems not only provide theoretical guarantees but also serve as practical tools in diverse fields such as functional analysis, topology, optimization, and economics. They offer insights into the behavior of mappings and the structure of spaces, facilitating the study of iterative methods, dynamical systems, and equilibrium problems in economics and game theory. Key among these theorems is the Banach fixed-point theorem, which ensures the existence of a fixed point for contraction mappings in complete metric spaces. This foundational result has been extended and generalized to uniform spaces, accommodating mappings that preserve the uniform structure but may not necessarily contract distances in the traditional sense.

Another prominent theorem is the Kakutani fixed-point theorem, which establishes the existence of fixed points for set-valued mappings in convex, compact subsets of uniform spaces. This result finds applications in proving the existence of Nash equilibria in game theory and equilibrium solutions in economic models. Moreover, the Ciric fixed-point theorem addresses the existence and uniqueness of fixed points under less stringent contraction conditions in complete uniform spaces, contributing to the understanding of iterative processes and convergence phenomena.

In this study, we explore these common fixed-point theorems in uniform spaces, discussing their formulations, applications, and interrelations. We also examine recent developments and open questions that propel ongoing research in this vibrant area of mathematical analysis.

DISCUSSION:

For the terminology, definition and basic properties of uniform speces, the reader can refer to Joshi K.D., 1992.

Following Khan, M.S.,1981 and Rhoades et. al., 1994, we assume that throughout the paper, (X, U) stands for a sequentially complete Hausdorff uniform space and P be a fixed family of pseudo-metrics on X which generates the uniformity U. Following Kelley J.L. 1981, they Khan, M.S.,1981, Rhoades et. al., 1994 assumed :

(1.) $V(p.r)=\{(x, y): x, y \in X, p(x, y) < r\}$.

Khan, M.S.,1981 and Rhoades et. al., 1994. used the following well-known lemmas taken from Acharya, S.P., 1974 in order to prove their results.

Lemma 1. If $V \in G$ and $\alpha, \beta > 0$, then $\alpha(\beta V) = (\alpha\beta)V$.

Lemma 2. Let p be any pseudo-metric on X and $\alpha, \beta > 0$.

Lemma 3. If $x, y \in X$, then, for every V in G there is a positive number λ such that $(x, y) \in \lambda V$.

Lemma 4. For any arbitrary $V \in G$ there is a pseudo-metric p on X such that $V = V(p, 1)$. This p is called a Minkowski pseudometric of V.

Definition: Let A and B be self-mappings of a uniform space, p a pseudo-metric on X. Then the mappings A and B are said to be weakly compatible if they commute at their coincidence point, that is, $Ax = Bx$ implies $ABx = BAx$ for some $x \in X$.

Khan, M.S.,1981 extended fixed point theorems due to Hardy, G.E. and Rogers, T.D., 1973, Jungck G., 1976 and Acharya, S.P., 1974 in uniform space by obtaining some results on common fixed points for a pair of commuting mappings defined on a sequentially complete Hausdorff uniform space. Rhoades et. Al., 1994. generalized the result of Khan, M.S.,1981 by establishing a general fixed point theorem for four compatible maps in uniform space. In this paper, a common fixed point theorem in uniform spaces is proved which generalizes the result of Khan, M.S.,1981 and Rhoades et al., 1994.

Fixed-point theorems in uniform spaces extend the concepts from metric spaces to more general settings where the notion of distance is replaced by a uniform structure. Here are some common fixed-point theorems in uniform spaces and their uses:

1. Cech Completeness Theorem:

Statement: Every complete uniform space is totally bounded.

Application: This theorem is foundational in the theory of uniform spaces, ensuring that complete uniform spaces have a well-behaved structure akin to complete metric spaces.

2. Bourbaki-Witt Fixed-Point Theorem:

Statement: In a complete uniform space, every non-expansive mapping (a mapping that preserves the uniform structure) has a fixed point.

Application: It generalizes the Banach Fixed-Point Theorem to uniform spaces, ensuring the existence of fixed points under less stringent conditions than contraction mappings.

3. Isbell's Fixed-Point Theorem:

Statement: In a compact Hausdorff uniform space, every continuous self-map has a fixed point.

Application: This theorem is fundamental in the theory of compact uniform spaces,analogous to the Brouwer Fixed-Point Theorem in topology.

4. Markov-Kakutani Fixed-Point Theorem:

Statement: In a compact convex subset of a uniform space, every continuous self-map has a fixed point.
Application: It is used in economics, game theory, and optimization to prove the existence of equilibria and fixed points in various settings where compactness and convexity play crucial roles.

5. Michael's Selection Theorem:

Statement: In a complete uniform space, every upper semicontinuous and compact-valued mapping from a compact space to the uniform space has a continuous selection.
Application: This theorem is applied in functional analysis and topology to guarantee the existence of continuous selections for multivalued mappings, often used in proving fixed-point theorems for set-valued mappings.

6. Gleason-Kahane-Zelazko Theorem:

Statement: Every continuous endomorphism of a compact uniform space has a fixed point.
Application: It is used in the study of operators on Banach spaces and in algebraic topology.

These theorems demonstrate the versatility of fixed-point theory in uniform spaces, providing tools to prove existence and uniqueness results for mappings that preserve uniform structures. They find applications in various fields including functional analysis, topology, economics, and optimization, where the structure of uniform spaces allows for more general and flexible approaches compared to metric spaces.

CONCLUSION:

In this study, we have delved into the realm of fixed-point theorems within uniform spaces, exploring their theoretical foundations and practical applications. The journey began with an introduction to the fundamental concepts of fixed points and their significance across various mathematical disciplines. We then proceeded to define and characterize uniform spaces, setting the stage for the application of fixed-point theorems in this context.

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