Indian Streams Research Journal
ISSN 2230-7850
Impact Factor : 3.1560(UIF)
Volume-4 | Issue-12 | Jan-2015
Available online at www.lbp.world

EXACT INHOMOGENEOUS SOLUTIONS

Dr. Amit Kumar Srivastava

Department of Physics , D.A.V. College, Kanpur , (U.P.), India.

Abstract

In an inhomogeneous metric which admits two spacelike commuting Killing vectors and has separable metric coefficients and obtained a general class of inhomogeneous solutions of Einstein field equations with nonthermalised perfect fluid as the source term.This idea arose out of the general result that under physically reasonable conditions of positivity of energy, causality, regularity etc., the initial singularity is inescapable in cosmology so long as we adhere to Einstein's equations and it may only be avoided by invoking quantum effects and or modifying Einstein general theory of relativity. In an inhomogeneous metric which admits two spacelike commuting Killing vectors and has separable metric coefficients and obtained a general class of inhomogeneous solutions of Einstein field equations with nonthermalised perfect fluid as the source term.

1.INTRODUCTION

Since the appearance of general theory of relativity the study of the Universe as whole has been on outstanding scientific subject. Until very recently, the exact solutions used for that study have been spatially homogeneous and isotropic distribution for its matter content being the FLRW (Friedman-Lemaitre-Robertson-Walker) models which admit at least a three-parameter group of isometrics and has been quite successful in describing the present state of the Universe. It is realised that the homogeneous and isotropic character of the spacetime may not be sustained at all scales especially for the early times. One of the main features of relativistic cosmology is the prediction of big bang singularity in the finite past. This idea arose out of the general result that under physically reasonable conditions of positivity of energy, causality, regularity etc., the initial singularity is inescapable in cosmology so long as we adhere to Einstein's equations and it may only be avoided by invoking quantum effects and or modifying Einstein general theory of relativity. It also shown, from the nonvanishing components of the Weyl tensor that the metric will be, in general, of Petrov type I at least in generic points. Very special cases could arise in which the Weyl tensor is of Petrov types D or O.

2. Key words: nonthermalised, inhomogeneous, spacelike, (Friedman-Lemaitre-Robertson-Walker) FLRW, hypersurface, nonthermalised, hypersurface

INTRODUCTION

The general inhomogeneous metric does not have any symmetry at all, but the complexity of Einstein equations is so high that some simplifications must be assumed. The simplest in homogeneous models are those with two spacelike commuting Killing vectors known as orthogonally transitive G_2 cosmologies as presented by (*Hewitt and Wainwright 1990*), (*Wainwright 1981*), (*Carmeli et al 1981*). Very few solutions of this type for perfect fluid are known up to now. The first class of solutions was given by (*Wainwright and Goode 1980*), and new metrics were later found by (*Feinstein and Senovilla 1989*), (*Senovilla 1990*) and (*Davidson 1991*). (*Senovilla 1990*) obtained new class of exact solutions of the Einstein equations without big bang singularity, representing cylindrically symmetric, inhomogeneous Universe with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. However, in order to study inhomogeneous epochs of the Universe, which apparently are necessary for the formation of large-scale structures, it is necessary to use exact inhomogeneous solutions to Einstein equations. The general inhomogeneous metric does not have any symmetry at all, but the complexity of Einstein equations is so high that some simplifications must be assumed. The simplest in homogeneous models are those with two spacelike commuting Killing vectors known as orthogonally transitive G_2 cosmologies as presented by (*Hewitt and Wainwright 1990*), (*Wainwright 1981*), (*Carmeli et al 1981*). Very few solutions of this type for perfect fluid are known up to now. The first class of solutions was given by (*Wainwright and Goode 1980*), and new metrics were later found by (*Feinstein and Senovilla 1989*), (*Senovilla 1990*) and (*Davidson 1991*). (*Senovilla 1990*) obtained new class of exact solutions of the Einstein equations without big bang singularity, representing cylindrically symmetric, inhomogeneous Universe with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. All the physical and geometrical invariants for it are finite and regular throughout the whole spacetime. Ruiz and (*Senovilla 1992*) have separated out fairly large class of singularity free models through a comprehensive study of general cylindrically symmetric metric with separable functions of r and t as metric coefficients. (*Dadhich, Tikekar and Patel 1993*) have obtained a link between the FLRW models and the singularity free family.

At very early times, matter in Universe is assumed to be in a highly dense and hot state. Subsequently, study of dissipative, effects in cosmology become very significant. Dissipative effects were studied in cosmology in the context of large entropy per baryon and isotropy of microwave background radiation by (*Misner 1968*), (*Caderni and Fabri (1978*). Cosmological models with heat flux have been investigated by several authors : (*Dang 1989*), (*Novello and Reboucas 1978*), (*Ray 1980*), (*Reboucas and Limma 1981*), (*Patel and Dadhich 1991*), (*Patel and Dadhich (1993),* (*Davidson 1993*).

In an inhomogeneous metric which admits two spacelike commuting Killing vectors and has separable metric coefficients and obtained a general class of inhomogeneous solutions of Einstein field equations with nonthermalised perfect fluid as the source term.

3. METHOD
Metric and Basic Equations

In the investigation of orthogonally transitive diagonal G_2 cosmologies as presented by (*Hewitt and Wainwright 1990*), geometrically this means that the spacetime admits two commuting space like Killing vector fields, both of which are hypersurface orthogonal and that the Einstein field equations are satisfied for an energy momentum tensor of nonthermalised perfect fluid. It is obvious that the metric of such spacetime may be put in the generalized Einstein-Rosen form.

1. $ds^2 = F_0\, dt^2 - F_1 dx^2 - F_2\, (F_2\, dy^2 + F_3^{-1}\, dz^2)$,

where the positive functions

2. $F_\alpha\ (\alpha, \beta\ =\ 0, 1, 2, 3)$,

depend on the coordinates t and x, the Killing vectors are $\partial/\partial y$ and $\partial/\partial z$ and the unit velocity vector of the fluid is

3. $\underline{u} \quad = \quad F_0^{1/2}\, dt.$

We shall consider only metrics of type (1) such that the functions F_α are separable i.e.

4. $F_\alpha\ =\ T_\alpha\ (t)\ \ X_\alpha\ (x).$

(*Riuz and Senovilla 1992*) have obtained that the metric for such spacetime assumes the form

5. $$ds^2 = T^{2m}\, F^2\left(dt^2 - H^2 dx^2\right) - T^{n+1} GP\, dy^2 - T^{1-n}\, GP^{-1}\, dz^2$$

where

$T = T\ (t)$
$F = F\ (x)$

6. $G = G\ (x)$
$P = P\ (x)$
$H = H\ (x)$

and m and n are constants. The coordinate are labelled as

7.
$x^o = t$
$x^1 = x$
$x^2 = y$
$x^3 = z$

The function H(x) will enable us to integrate the field equations in terms of elementary functions in some occasions. Let us introduce the orthonormal tetrad

8.
$$\begin{aligned} \theta^{o} &= T^{m} F\, dt \\ \theta^{1} &= T^{m} FH\, dx \\ \theta^{2} &= T^{(1+n)/2} \sqrt{GP}\; dy \\ \theta^{3} &= T^{(1-n)/2} \sqrt{G/P}\, dz \end{aligned}$$

The nonvanishing components of the Ricci tensor for the metric (2.5) in the above tetrad frame assume the form

9.
$$R_{00}\, T^{2m} F^{2} = \frac{1}{H^{2}}\left[\frac{F''}{F} - \frac{F'^{2}}{F^{2}} - \frac{F'H'}{FH} + \frac{F'G'}{FG}\right] + (m+1)\frac{\ddot{T}}{T} + \frac{1}{2}(n^{2} - 1 - 4m)\frac{\dot{T}^{2}}{T^{2}}$$

10.
$$R_{11} T^{2m} F^{2} = \frac{1}{H^{2}}\left[\frac{G''}{G} \quad \frac{1}{2}\frac{G'^{2}}{G^{2}} + \frac{1}{2}\frac{P'^{2}}{P^{2}} + \frac{F''}{F} - \frac{F'G'}{FG} - \frac{H'G'}{HG} - \frac{F'^{2}}{F^{2}} - \frac{F'H'}{FH}\right] - m\frac{\ddot{T}}{\mathrm{T}}$$

11.
$$R_{22} T^{2m} F^{2} = \frac{1}{H^{2}}\left[\frac{1}{2}\left(\frac{G''}{G} - \frac{P''}{P}\right) - \frac{1}{2}\frac{P'^{2}}{P^{2}} + \frac{1}{2}\frac{G'P'}{GP} - \frac{1}{2}\frac{H'G'}{HG} + \frac{1}{2}\frac{P'H'}{PH}\right] - \frac{1}{2}(1+n)\frac{\ddot{T}}{T}$$

12.
$$R_{33} T^{2m} F^{2} = \frac{1}{H^{2}}\left[\frac{1}{2}\left(\frac{G''}{G} + \frac{P''}{P}\right) - \frac{1}{2}\frac{P'^{2}}{P^{2}} + \frac{1}{2}\frac{G'P'}{GP} - \frac{1}{2}\frac{H'G'}{HG} - \frac{1}{2}\frac{P'H'}{PH}\right] - \frac{1}{2}(1-n)\frac{\ddot{T}}{T}$$

13.
$$R_{10} T^{2m} F^{2} = \frac{1}{2H}\frac{\dot{T}}{T}\left[(1-2m)\frac{G'}{G} + n\frac{P'}{P} - 2\frac{F'}{F}\right]$$

where does and primes denote derivatives with respect to t and x, respectively.
The energy momentum tensor for a nonthermalizaed perfect fluid reads –

14.
$$T_{\alpha\beta} = (\rho + p)\, u_{\alpha} u_{\beta} - p\, g_{\alpha\beta} + (q_{\alpha} u_{\beta} + q_{\beta} u_{\alpha})$$

15. $u_\alpha u^\alpha = 1$
16. $u_\alpha q^\alpha = 0$

where ρ and p be the energy density and fluid pressure of distribution. u^α denotes unit time like flow vector of the fluid and q^α stands for space like heat flow vector orthogonal to u^α .

The components of vector u^α and q^α in tetrad frame assume the form

17. $u^\alpha = (1, 0, 0, 0)$
18. $q_\alpha = (0, q, 0, 0)$

where q is function of coordinates and depends on the Einstein field equations.

19. $$R_{\alpha\beta} = -8\pi\left\{(\rho + \mathrm{p})u_\alpha u_\beta - \frac{1}{2} g_{\alpha\beta}(\rho - p)\right\} - 8\pi\left(q_\alpha u_\beta - u_\alpha u_\beta\right)$$

In view of eqs. (17) - (19) one obtains

20. $R_{11} = 4\pi g_{11} (\rho - p) = -4\pi(\rho - p)$
21. $R_{22} = 4\pi g_{22} (\rho - p) = -4\pi(\rho - p)$
22. $R_{33} = 4\pi g_{33} (\rho - p) = -4\pi(\rho - p)$
23. $R_{00} = -8\pi\{(\rho + p) - ½ g_{00} (\rho - p)\}$
$= -8\pi(\rho/2 + \frac{3p}{2}) = -4\pi(\rho + 3p)$

24. $R_{10} = -8\pi q$

showing that the following relations hold

25. $R_{11} = R_{22} = R_{33}$
26. $8\pi\rho = -½(3R_{22} + R_{00})$
27. $8\pi p = ½(R_{22} - R_{00})$

In view of eq. (24) it is obvious that $R_{10} \neq 0$ showing that the spacetime may sustain presence of heat flow. If $R_{10} = 0$ i.e. $q = 0$ one may recover the perfect fluid i.e.

28. $F^2 = G^{1-2m} P^n$

In our case $R_{10} \neq 0$, hence we select

29. $F^2 = G^{2d} P^{2\lambda}$

In view of equation (25), one obtains

30. $$\frac{1}{H^2}\left[\frac{G'P'}{GP}-\frac{P'H'}{PH}+\frac{P''}{P}-\frac{P'^2}{P^2}\right]=n\frac{\ddot{T}}{T},$$

31.
$$\frac{1}{H^2}\left[-\frac{P''}{P}-\frac{G''}{G}+\frac{G'^2}{G^2}-2\frac{F''}{F}-\frac{G'P'}{GP}+2\frac{F'G'}{FG}+\frac{H'G'}{HG}+\frac{P'H'}{PH}+2\frac{F'^2}{F}+2\frac{F'H}{FH}\right]$$

$$= \frac{\ddot{T}}{T}(1-n-2m)$$

The left hand side of eqs. (30) – (31) are functions of x only whereas the right hand sides are functions of t only. Hence, both sides of respective equations must be equal to a separation constant, different for different equations. Hence, one obtains

32. $$\frac{\ddot{T}}{T}=\in a^2,\ \in=0,\pm1,\ \mathrm{a}=\mathrm{constant}$$ implying that

33. $$T(t) = \begin{cases} A\,\mathrm{Cosh}\,(at) + B\,\sinh\,(at), & \in = 1 \\ At + B, & \in=0 \\ A\,\mathrm{Cos}\,(at) + B\,\mathrm{Sin}\,(at), & \in = -1 \end{cases}$$

where A and B are arbitrary constants of integration. In view of eqs. (29) and (32), the eqs. (30) and (31) assume the form

34. $$\alpha\beta+\alpha' - \alpha\frac{H'}{H} = \in n\, a^2\, H^2$$

35. $$\alpha^2 - 4\lambda\alpha\beta+(2d+1)\beta' - 2\beta^2 d - (2d+1)\beta\frac{H'}{H}$$

$$= (2m-1-2n\lambda) \in a^2 H^2$$

Where

36. $\beta = G'/G$

37. $\alpha = P'/P$

One may obtain the physical variables p, ρ, q in view of equations 29 – 34

38. $$32\pi\ p\,F^2 T^{2m} = \in a^2\left(4m - n^2 - 3\right) + \left(4m + 1 - n^2\right)\left(\frac{\dot{T}^2}{T^2} - \in a^2\right) + \frac{X}{H^2}$$

39. $$32\pi\,F^2\,T^{2m} = \in a^2\left\{2m + 2 - n^2 + 2\lambda\,n - \frac{(2d-3)}{(2d+1)}(1 - 2m + 2n\lambda)\right\}$$
$$+\ (4m+1-n)\left(\frac{\dot{T}^2}{T^2} - \in a^2\right) + \left(\frac{2d - 3}{2d + 1}\right)\ \frac{X}{H^2}$$

40. $$16\pi\,qF^2\,T^{2m} = -\frac{\dot{T}}{T} - \frac{1}{H}\left\{(1 - 2m - 2d)\beta + (n - 2\lambda)\alpha\right\}$$

where

41. $X = (1+4d)\beta^2 - \alpha^2 + 4\lambda\alpha\beta$

and we have taken always that

42. $2d + 1 \neq 0$

In the view of eqs. (38) and (39), one may easily obtain

43. $$\rho = \left(\frac{2d-3}{2d+1}\right)p + \frac{1}{16\pi\,F^2\,T^{2n}}\left\{\frac{2(4m+1-n^2)}{d2+1}\left(\frac{\dot{T}^2}{T^2} - \in a^2\right)\right.$$
$$\left. + \in a^2\,\frac{(2d + 2m - 1 + 2\lambda\ n - n^2)}{2(2d+1)}\right\}$$

The matter distribution will satisfy an equation of state

44. $\rho = K\,p$

if one of the following set of condition is satisfied :

45. $\in = 0,\ \ 4m+1 = n^2$

46. $2d + 2m - 1 + 2\lambda n - n^2 = 0,\ \ 4m+1 = n^2$,

47. $2d + 2m - 1 + 2\lambda n - n^2 = 0,\ \ \frac{\dot{T}^2}{T} = \in a^2$

Where

48. $$K = \frac{2d - 3}{2d + 1} \neq 1$$

However, corresponding to stiff fluid, the parameters may be selected appropriately so that the matter distribution satisfies equation of state $\rho = p$. For $\rho = Kp$ and K as positive, $d > 2$ must be taken. Here q be the parameter which implies the presence of heat flow with matter and will vanish if anyone of the following conditions is satisfied:

49. $2d = 1 - 2m$, $\alpha = 0$,
50. $n = 2\lambda$, $\beta = 0$,
51. $\alpha = 0$, $\beta = 0$,
52. $2d = 1 - 2m$, $n = 2\lambda$

Hence, we have investigated the heat flow generalisation of the general class of inhomogeneous perfect fluid solutions of (*Ruiz and Senovilla 1992*) when $2d \neq 1 - 2m$ and $n \neq 2\lambda$; $\alpha \neq 0$ and $\beta \neq 0$.

4. KINEMATICAL PARAMETERS

The kinematical properties of the velocity vector given by eq. (17) in the metric by eq. (5), a straightforward calculation leads to the following expression for the expansion θ, rotation w, shear σ, and acceleration a_i of the fluid

53. $$0 = \left(\frac{m+1}{F}\right)\frac{\dot{T}}{T^{m+1}}$$

54. $w = 0$

55. $$\sigma^2 = \left[\frac{(1-2m)^2 + 3n^2}{6F^2}\right]\frac{\dot{T}^2}{T^{2m+2}}$$

56. $a_i = (0, -d\beta, -\lambda\alpha, 0, 0)$

The ratio of σ to θ is constant.

57. $$\frac{\sigma}{\theta} = \frac{\sqrt{\frac{(1-2m)^2 + 3n^2}{6F}}\ \frac{\dot{T}}{T^{m+1}}}{\left(\frac{m+1}{F}\right)\frac{\dot{T}}{T^{m+1}}} + \frac{\sqrt{(1-2m)^2 + 3n^2}}{\sqrt{6}\,(m+1)}$$

showing that there is no possibility of the models to get isotropised at some later time.
For $n = 0$ and $2m = 1$, one obtains

58. $$\theta = \frac{3}{2F}\frac{\dot{T}}{F^{3/2}}$$

59. $\sigma = 0$

Hence, we have obtained cylindrically symmetric models for non-thermalised shear-free perfect fluid distribution with

60. $$q = \frac{+\dot{T}}{16\pi T^2 F^{-2} H}\left[+2\,d\beta + 2\lambda\,\alpha\right]$$

$$= \frac{\dot{T}}{8\pi T^2 F^{-2} H}\left[\beta d + \lambda\,\alpha\right]$$

5. The nonvanishing Components of the Weyl Tensor

Performing the computation in the null tetrad

61. $$\underline{k} = \frac{1}{\sqrt{2}}\left(\theta^0 - \theta^1\right),$$

62. $$\underline{l} = \frac{1}{\sqrt{2}}\left(\theta^0 + \theta^1\right),$$

63. $$\underline{m} = \frac{1}{\sqrt{2}}\left(\theta^2 + i\theta^3\right)$$

one obtains

64. $$\psi_2 = \frac{1}{6}T^{-2m}F^{-2}\left[\frac{1}{H^2}\left(\frac{F''}{F} - \frac{F'^2}{F^2} - \frac{F'H'}{FH} - \frac{1}{2}\beta' + \frac{1}{2}\frac{H'}{H}\beta - \frac{1}{2}\alpha^2\right)\right.$$
$$\left. + \frac{(2m+n^2-1)}{2}\frac{\dot{T}^2}{T^2} + \frac{1-2m}{2} \in a^2\right]$$

65. $$\psi_0 - \psi_4 = \frac{T^{-2m}\,F^{-2}}{H}\,\frac{\dot{T}}{T}\left[\left(\frac{2m-1}{2}\right)\alpha + n\frac{F'}{F}\frac{n}{2}\beta\right]$$

66. $$\psi_0 + \psi_4 = T^{-2m}\,F^{-2}\left[\frac{1}{H^2}\left(\alpha\frac{F'}{F} - \frac{1}{2}\alpha\,\beta + \frac{1}{2}\alpha\frac{H'}{H} - \frac{1}{2}\alpha'\right)\right.$$

$$+ mn\frac{\dot{T}^2}{T^2} - \frac{n}{2} \in a^2 \Bigg]$$

It is obvious that the eqs. (64) – (66) may be further simplified by using eq. (29) and the main eqs. (34) and (35) so that there would not appear any derivative of the functions α and β.

6. CONCLUSION

We have obtained heat flow generalisation of the general class of inhomogeneous perfect fluid solutions of (*Ruiz and Senovilla 1992*) when $2d \neq 1-2m$ and $n \neq 2\lambda$; $\alpha \neq 0$ and $\beta \neq 0$. In our model the ratio of σ to θ is a constant showing that there is no possibility of the models to get isotropised at some later time. In this set up $2m = 1$ and $n = 0$, we have investigated cylindrically symmetric models describing distribution of nonthermalised shear-free perfect fluid. It also shown, from the nonvanishing components of the Weyl tensor that the metric will be, in general, of Petrov type I at least in generic points. Very special cases could arise in which the Weyl tensor is of Petrov types D or O.

REFRENCES

1. Caderni, N. and Fabri, R. (1978), Nuovo Cimento B44, 288.
2. Carmeli, M., Ch. Charach, and Malin, S. (1981), Phys. Rep.79, 80
3. Davidson, W. (1991), J. Math. Phys. 32, 1560.
4. Davidson, W. (1993). Class Q. Grav. 10, 1843
5. Deng, Y. (1989), GRG 21, 503.
6. Feinstein, A. and Senovilla, J.M.M. (1989), Class Q. Grav. 6, L 89.
7. Hewitt, C.G. and Wainwright, J. (1990), Class Q. Grav. 7, 2295.
8. Patel, L.K. and Dadhich, N.K. (1991), Astrophys. J. 401, 433.
9. Patel, L.K. and Dadhich, N.K. (1993), Classw Q. Grav.10,L L85.
10. Ray, D. (1980), J.Math. Phys. 21, 2797.
11. Reboucas, M.J. and de Limma, J.A.S. (1981), J. Math. Phys. 22, 2699.
12. Senovilla, J.M.M. (1990), Phys. Rev. Lett. 64, 2219.
13. Wainwright, J. (1981), J. Phys. A14, 1131.
14. Wainwright, J. and Goode, S. We. (1980), Phys. Rev. D22,1906.