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LEMMA AND THEOREM WHICH HELP US TO FIND A REPRESENTATION FOR A SOLENOIDAL VECTOR FIELD IN CARTESIAN CO-ORDINATES IN TERMS OF TWO SCALAR FUNCTIONS.

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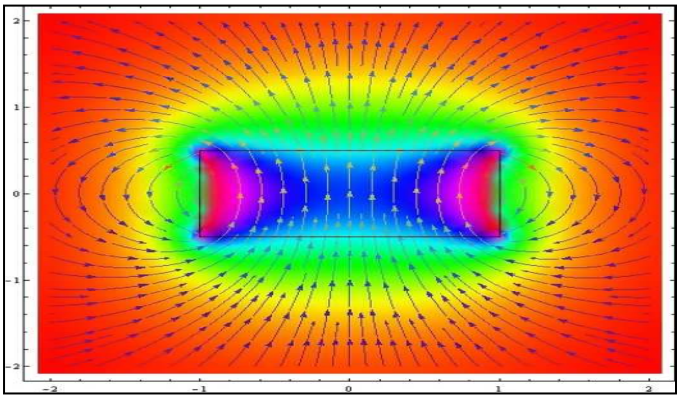
Abstract:
In many physical situations, the governing equations of motion are usually given as a vector equation where the quantity of physical interest may be a vector, like the fluid velocity in fluid flows. However, it has been observed that when these quantities are expressed in terms of a scalar field, the scalar satisfies a much simpler partial differential equation than the given vector.

KEYWORDS : *Lemma and theorem , quantity of physical interest , velocity potential.*

INTRODUCTION :

A simple example is the role played by a velocity potential in irrotaticnal flows or a stream function in two dimensional, incompressible, inviscid flows. Such a representation of vector fields using scalar functions is, in particular, useful in boundary value problems when the boundary conditions reduce to simple



relations in terms of these scalars.

In 1967, Chadwick and Trowbridge [6] showed that any divergence free vector field V can be expressed as

$$V = \mathrm{Curl}\,\mathrm{Curl}(rA) + \mathrm{Curl}(rB),$$

where A and B are scalar functions on any bounded annular domain

$$S = \{(r,\theta,\varphi) : r_1 \le r \le r_2, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\},$$

where $(r,\theta,\varphi)$are spherical polar co-ordinates.

This result was found to be extremely useful in the context of Stokes equations where the velocity vector is solenoidal as it gives rise to a complete general solution of Stokes equations [7], which in turn proves to be very convenient in problems involving spherical boundaries, since the scalar functions occurring in it satisfy simple partial differential equations which can be solved easily. Moreover, the boundary conditionrexpressed in terms of these scalar functions are of a very simple form.

Chadwick and Trowbridge [6] observed that the results can be extended to an infinitedomain (r2oo) with boundedness conditions On V. This extension was used in [7] to prove the completeness of a certain solution of Stokes equations. The proof of this extension was given by Padmavathi and Amaranath [9]. This representation may not be convenient to use for plane boundaries. In this chapter, a new representation is given for divergence free (solenoidal) vector fields which is useful in problems dealing with plane boundaries. By making use of this representation, we establish the completeness of some general solutions of Stokes, Brinkman and Oseen equations proposed for plane boundaries in a later chapter. Despite the similarity in the structure of the solution in both the spherical [7] and plane geometries, the proof of completeness of the solution in the plane boundaries case is not obtained by merely mimicking the proof given in [7] and requires completely a different approach altogether.

2.2Solenoidal Vector Fields in Spherical Polar Co-ordinates

We now discuss some results given in [9] which establish the completeness of certain solutions of Stokes and Brinkman equations in infinite domains involving spherical bound-aries. The following lemma and theorem for infinite domains are due to Padmavathi and Amaranath [9].

Lemma 1: Let Z $\in C^2$ on S_1, where S_1 is given by

$$S_1 = \{r, \theta, \varphi) : r \geq r_1 > 0\}$$

If Z satisfies

$$r \bullet Z = 0, \qquad (2.2.1)$$
$$r \bullet \mathrm{Curl} Z = 0, \qquad (2.2.2)$$
$$\Delta \bullet Z = 0, \qquad (2.2.3)$$

on S_1, then

$$Z = 0, \qquad (2.2.4)$$

ons$_l$.

Theorem 1: If V $\in C^2$ on S_1 and satisfies $\Delta \bullet V = 0$, then we can find scalar functions A and B such that on S_1, V can he represented as

$$V = \mathrm{Curl\ Curl}(rA) + \mathrm{Curl}(rB), \qquad (2.2.5)$$

where A and B are solutions of the following equations

$$LA = —r \bullet V, \qquad (2.2.6)$$
$$LB = —r \bullet \mathrm{Cur1}V, \qquad (2.2.7)$$

where L is the transverse part of the Laplace operator except for the factor $1/r^2$ in spherical polar co-ordinates (r, θ, φ). From the theorem discussed above, it is possible to express the velocity vector which is solenoidal in terms of two scalars A and B. This representation is used in Chapter-3 to prove the completeness of the solutions of homo¬geneous and non-homogeneous unsteady Stokes equations. In the next section, we shall discuss the representation of solenoidal vector fields in cartesian co-ordinates.

2.3 Solenoidal Vector Fields in Cartesian Co-ordinates

We state a lemma and theorem which enable us to find a representation of a solenoidal vector field in cartesian co-ordinates in terms -of certain scalar functions and also determine the partial differential equations satisfied by the scalars themselves. The proofs of the lemma and theorem given here cannot be obtained by merely emulating the proof given in the case of spherical boundaries stated in the previous section.

Lemma 2 : If Z E $C^2(IR^3)$ and satisfies

$$\hat{k} \bullet z = 0, \qquad (2.3.1)$$
$$\hat{k} \bullet \mathrm{Curl}Z = 0, \qquad (2.3.2)$$
$$\Delta \bullet Z = 0, \qquad (2.3.3)$$

then Z can be expressed in the form

$$Z = \mathrm{Cur1}(kB_1), \qquad (2.3.4)$$

Where

$$LB_1 = 0 \qquad (2.3.5)$$

And

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (2.3.6)$$

Proof: Let

$$\mathbf{Z} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}. \quad (2.3.7)$$

From equation (2.3.1), w = 0. Consider the following partial differential equations

$$B_{lx} = -u \quad (2.3.8)$$
$$Bly = u. \quad (2.3.9)$$

From (2.3.3), since

$$u_x - u_y = 0, \quad (2.3.10)$$

the above system (2.3.8, 2.3.9) is consistent and there exists a scalar function B1 which satisfies (2.3.4) or (2.3.8, 2.3.9). From equation (2.3.2), we have

$$v_x - u_y = 0. \quad (2.3.11)$$

Hence using equations (2.3.8) and (2.3.9), we get

$$B_{1xx} + B_{lyy} = 0, \text{ i.e., } LB_1 = 0, \quad (2.3.12)$$

which is the same as equation (2.3.5).

Theorem 2: If $V \in C^2(IR^3)$ and satisfies $\Delta \bullet \nabla = 0$, then we can find scalar functions A and B such that

$V = \text{Curl Curl}(\hat{k}A) + \text{Curl}(\hat{k}B)$,
$= (A_{xz} + B_y)\,\hat{\imath} + (A_{yz} - B_x)\,\hat{\jmath} - LA\hat{k}$. (2.3.13)

Proof : Let A and B_2 be the solutions of

$LA = -\hat{k} \bullet V$, (2.3.14)
$LB_2 = -\hat{k} \bullet \text{CurlV}$, (2.3.15)

respectively, where L is given in (2.3.6). For any fixed z, if f(x) denotes the right hand sides of equations (2.3.14) or (2.3.15) where X = (x, y) and if we assume that f satisfies the following conditions in the (x, y) plane :

1. $|\, f(\mathbf{x}) \,|\, log(1 + |\, \mathbf{x} \,|)$ is integrable,
2. f (x) is Holder continuous of exponent α for $0 < \alpha < 1$,

then the solutions of equations (2.3.14) and (2.3.15) exist and are twice continuously differentiable and the second derivative of the respective solutions are Holder continuous of the same exponent α [18]. We assume in the rest of the thesis that V is such that the right hand sides of equations (2.3.14) and (2.3.15) satisfy the above conditions. Define

$$Z = V - \text{Curl Curl}(\hat{k}A) - \text{Cur-1}(\hat{k}B_2). \quad (2.3.16)$$

Observe that

$$\hat{\mathbf{k}} \cdot \mathbf{Z} = \hat{\mathbf{k}} \cdot \mathbf{V} + LA = 0 \text{ (from (2.3.14))}, \quad (2.3.17)$$
$$\hat{\mathbf{k}} \cdot \mathrm{Curl}\mathbf{Z} = \hat{\mathbf{k}} \cdot \mathrm{Curl}\mathbf{V} + LB_2 = 0 \text{ (from (2.3.15))}, \quad (2.3.18)$$
$$\nabla \bullet Z = 0. \quad (2.3.19)$$

Using conditions (2.3.17)-(2.3.19), it follows from Lemma 2 that such a Z can be expressed
As

$$Z = \mathrm{Curl}(\hat{k}B_1), \quad (2.3.20)$$

Where

$$LB_1 = 0. \quad (2.3.21)$$

Hence we observe further that

$$\mathrm{Curl}(\hat{\mathbf{k}}B_1) = \mathbf{V} - \mathrm{Curl}\ \mathrm{Curl}(\hat{\mathbf{k}}A) - \mathrm{Curl}(\hat{\mathbf{k}}B_2), \quad (2.3.22)$$

Or

$$V = \mathrm{Curl}\ \mathrm{Curl}(\hat{k}A) + \mathrm{Cur1}(\hat{k}B_2). \quad (2.3.13)$$

Where

$$B = B_1 + B_2, \quad 2.3.23)$$

and therefore from (2.3.15) and (2.3.21)

$$LB = -\hat{k} \bullet \mathrm{Cur1}V. \quad (2.1724)$$

Hence the theorem.

The representation given for the velocity vector in (2.3.13) is used in the Chapter-4, to discuss complete general solutions of Stokes, Brinkman and Oseen equations in carte-sian co-ordinates. In particular, we discuss the Stokes flow in the presence of a plane boundary for both rigid as well as shear-free boundary conditions. It is observed here that the representation given in (2.3.13) is very simple to use as the boundary conditions formulated in terms of A and B assume a very simple form.

In the next chapter we discuss the complete general solutions of homogeneous and non-homogeneous unsteady Stokes equations and their applications.

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