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BOUNDS ON DOMINATION IN ZERO DIVISOR GRAPH

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Abstract:

Let R be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. The zero divisor graph of a ring is the graph(simple) whose vertex set is the set of non zero zero divisor, and an edge is drawn between two distict vertices if their product is zero. For a graph $\Gamma(Z_n)$, a set S⊆ $V(\Gamma(Z_n)$ is a dominating set of $\Gamma(Z_n)$, if every vertex in $V(\Gamma(Z_n) - S$ is adjacent to some vertex in S. The minimum Cardinality of vertex in such a set is called the domination number of $\Gamma(Z_n)$ and is denoted by $\gamma(\Gamma(Z_n))$. In this paper many bounds on $\gamma(\Gamma(Z_n))$ were obtained in terms of the elements of $\Gamma(Z_n)$. Also its relations with other domination parameters were obtained.

Keywords: Zero divisor graph/Domination.

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INTRODUCTION

Let R be a commutative ring and let Z(R) be its set of zero divisor. The zero-divisor graph of a ring is the simple graph whose vertex set is the set of non-zero zero divisors and an edge is drawn between two distinct and an edge is drawn between two distinct vertices if their product is zero. Throughout this paper, we consider the commutative ring by R and zero divisor graph by $\Gamma(Z_n)$. The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [2], where he was mainly interested in colourings. The zero-divisor graph is very useful of find the algebraic structures and properties of ring. In this paper many bounds on $\gamma(\Gamma(Z_n))$ were obtained in terms of the element of $\Gamma(Z_n)$. For notation and graph theory terminology, we in general follow [3], [4] and the structure of the zero divisor-graph is [1], [2]. In this paper, all the graph considered here are simple, finite, non-trivial, undirected and connected. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $\Gamma(Z_n)$ espectively. In general we use $< X >$ to denote the sub graph induced by the set of vertices X and N(v) and N[v] denote the open and closed neighbourhood of a vertex v. The minimum (maximum) degree among the vertices of $\Gamma(Z_n)$ is denoted by $\delta(\Gamma(Z_n))(\Delta(\Gamma(Z_n))$. Further $\beta_0(\Gamma(Z_n))$ represent the vertex independence number of $\Gamma(Z_n)$ and the term $\alpha_0(\Gamma(Z_n))$ denote the minimum number of vertices cover of $\Gamma(Z_n)$. The domination theory is very useful in communication network, computer science and so on.

Theorem 1: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) \leq \beta_0(\Gamma(Z_n))$.

Proof: Let S be an independent set of vertices in $\Gamma(Z_n)$ such that $|S| = \beta_0(\Gamma(Z_n))$. Then $\Gamma(Z_n)$ contains no large independent set. This means that every vertex v in is adjacent to at least one vertex of S. Therefore S is a dominating set. Also $D = \Gamma(Z_n) - S$ is a minimal dominating set of $\Gamma(Z_n)$. It follows that $|D| \leq |S|$. Thus $\gamma(\Gamma(Z_n)) \leq \beta_0(\Gamma(Z_n))$.

Theorem 2: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) \leq p - \alpha_0(\Gamma(Z_n))$.

Proof: Let $A = \{v_1, v_2, - - -, v_i\}$ where $d(v_i) \geq 2, 1 \leq i \leq n$ be the set of vertices which covers all the edges of $\Gamma(Z_n)$ such that $|A| = \alpha_0(\Gamma(Z_n))$. Let $D = \{v_1, v_2, - - -, v_k\}$ be the minimal dominating set of $\Gamma(Z_n)$. It follows that $D \subseteq A$. Clearly $|D| = p - |A|$ and hence $\gamma(\Gamma(Z_n)) \leq p - \alpha_0(\Gamma(Z_n))$.

Theorem 3: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\alpha_0(\Gamma(Z_n)) + \beta_0(\Gamma(Z_n) = p$.

Proof: Let S be a maximal independent set of vertices in $\Gamma(Z_n)$ such that $|S| = \beta_0(\Gamma(Z_n))$. Then there are no edges in induced sub graph $< S >$ of $\Gamma(Z_n)$. So every edge is incident to at least one vertex of $V(\Gamma(Z_n)) - S$. If $V(\Gamma(Z_n)) - S$ covers all the edges then there are no edges in $< S >$. It follows that $|V(\Gamma(Z_n)) - S| = \alpha_0(\Gamma(Z_n))$. Clearly

$\alpha_0(\Gamma(Z_n)) + \beta_0(\Gamma(Z_n) = |V(\Gamma(Z_n)) - S| + |S| = V(\Gamma(Z_n)) = p$.

Theorem 4: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) \leq p - \Delta(\Gamma(Z_n))$.

Proof: Let $A = \{v_1, v_2, - - -, v_i\} \subseteq V(\Gamma(Z_n))$ such that $deg(v_i) \geq 2, 1 \leq i \leq k$. Then there exists at least one vertex $v \in V$ such that $deg(v_i) = \Delta(\Gamma(Z_n))$. Hence $V(\Gamma(Z_n)) - N(v)$ is a dominating set. Let $D = \{v_1, v_2, - - -, v_n\}$ be a minimal dominating set of $\Gamma(Z_n)$. It follows that, $|D| - |V(\Gamma(Z_n)) - N(v)|$, which gives $\gamma(\Gamma(Z_n)) \leq p - \Delta(\Gamma(Z_n))$.

Theorem 5: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\left\lceil \frac{p}{1+\Delta(\Gamma(Z_n))} \right\rceil \leq \gamma(\Gamma(Z_n))$.

Proof: Let D be a γ set of $\Gamma(Z_n)$ and each vertex dominates at most itself and $\Delta(\Gamma(Z_n))$ other vertices, so $\left\lceil \frac{p}{1+\Delta(\Gamma(Z_n))} \right\rceil \leq \gamma(\Gamma(Z_n))$.

Theorem6: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) \leq \frac{p}{2}$.

Proof: Let $D = \{v_1, v_2, - - -, v_k\}$ be a minimal dominating set of G. By Theorem 4, $V(\Gamma(Z_n)) - D$ is a dominating set of $\Gamma(Z_n)$. But $|D| < p$ and $|V(\Gamma(Z_n)) - D| < p$. Hence $\gamma(\Gamma(Z_n)) \leq \min\{|D|, |V(\Gamma(Z_n)) - D|\} \leq \frac{p}{2}$.

Theorem 7: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) + diam((\Gamma(Z_n)) \leq p$.

Proof: Let $I = \{e_1, e_2, - - -, e_k\}$ subset of $E(\Gamma(Z_n)$ be the set of edges which constitutes the longest path between any two distinct vertices of $\Gamma(Z_n)$ such that $|I| = diam(\Gamma(Z_n))$.Let $D = \{v_1, v_2, - - -, v_k\}$ be any minimal dominating set of $\Gamma(Z_n)$. But $|D| \cup |I| \leq p$. It follows that $\gamma(\Gamma(Z_n)) + diam((\Gamma(Z_n)) \leq p$.

Theorem 8: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) \leq p - \delta(\Gamma(Z_n)) + 1$.

Proof: Let $D = \{v_1, v_2, - - -, v_k\}$ be a minimal dominating set in $\Gamma(Z_n)$. Then there exists at least one vertex v of minimum degree$\delta(\Gamma(Z_n)) \in V(\Gamma(Z_n))$. Since at least one vertex of D is adjacent to a vertex of minimum degree. Then $|D| \leq |V| - \delta(\Gamma(Z_n)) + 1$. Which gives $\gamma(\Gamma(Z_n)) \leq p - \delta(\Gamma(Z_n)) + 1$.

Theorem 9: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\chi(\Gamma(Z_n)) \leq \Delta(\Gamma(Z_n)) + 1$. Equality form if $\Gamma(Z_n)$ is a complete graph.

Proof: Select any vertex v of the graph $\Gamma(Z_n)$. Colour v with colour 1. Color all other vertices adjacent to those that have been colored with 2 using color 1. Continue the process till all the vertices in $\Gamma(Z_n)$ has been colored. Now we find that all vertices in $\Gamma(Z_n)$ at odd distance from v have colored with 2. While v and vertices at even distance from v have colour 1. Thus $\Gamma(Z_n)$ has been properly colored with two colors. Let u is adjacent to $N(u)$ vertices such that $\Delta(\Gamma(Z_n)) = N(u)$ with two colors. It follows that $\chi(\Gamma(Z_n)) \leq \Delta(\Gamma(Z_n)) + 1$. If graph $\Gamma(Z_n)$ is a complete then each pair of distinct vertices is joined by an edge so that in this graph we required by number of vertices is equal to number of colors and in complete graph each vertex of degree is $p - 1$. Hence $\chi(\Gamma(Z_n)) = p = \Delta(\Gamma(Z_n)) + 1$.

Theorem 10: Let R be a finite commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. Then $\gamma(\Gamma(Z_n)) + \chi(\Gamma(Z_n)) \leq p + 1$.

Proof: By Theorem4, $\gamma(\Gamma(Z_n)) \leq p - \Delta(\Gamma(Z_n))$ and by Theorem9, $\chi(\Gamma(Z_n)) \leq \Delta(\Gamma(Z_n)) + 1$. Hence $\gamma(\Gamma(Z_n)) + \chi(\Gamma(Z_n)) \leq p - \Delta(\Gamma(Z_n)) + \Delta(\Gamma(Z_n)) + 1 \leq p + 1$.

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