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ON SOME NEW SIMULTANEOUS INTEGRAL AND INTEGRO-DIFFERENTIAL INEQUALITIES

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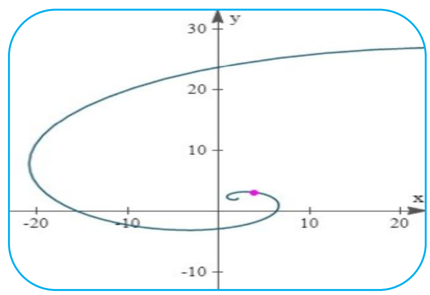
ABSTRACT:
This paper explores novel simultaneous integral and integro-differential inequalities through an analytic approach. The study aims to establish explicit bounds for the solutions of specific simultaneous differential, integral, integro-differential equations, and SIS models (of Gonorrhea type).



KEYWORDS: *simultaneous integral and integro-differential , integral, integro-differential equations.*

1. INTRODUCTION

The study of differential, integral, and integro-differential equations holds a pivotal role in advancing scientific knowledge. Inequalities play a crucial role in the analysis of such equations, providing clear bounds for the functions involved. These inequalities have gained widespread acceptance, underscoring the need for the evaluation of new inequalities to enhance their applications in analyzing various classes of integral and differential equations.

Over the past few years, numerous authors have documented integral inequalities [1-17], emphasizing their significance in the realm of integral and differential equations. Notably, these inequalities play a vital role in the expansion of the understanding of such equations. In 1977, D. E. Greene [8] introduced key simultaneous integral inequalities, while B.G. Pachpatte [16] later established their effective generality. These simultaneous integral inequalities prove instrumental in analyzing diverse problems within the theory of systems of differential and integral equations.

This paper introduces innovative simultaneous integral and integro-differential inequalities, serving as powerful tools for exploring qualitative properties such as existence, uniqueness, boundedness, stability, and asymptotics of solutions in simultaneous integral and integro-differential equations.

In this article, $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.

2. Integral Inequalities

In this section, we state and prove integral inequalities.

Theorem 2.1 *Suppose* $u, v, h_i (i = 1\ldots,6)$ *non-negative continuous functions on* $\mathbb{R}_+$ *and* k_1, k_2 *non-negative constants. If*

$$u(t) \le k_1 + \int_0^t h_1(\omega)u(\omega)ds + \int_0^t h_2(\omega)v(\omega)d\omega + \int_0^t h_3(\omega)u(\omega)v(\omega)d\omega \tag{2.1}$$

$$v(t) \le k_2 + \int_0^t h_4(\omega)u(\omega)d\omega + \int_0^t h_5(\omega)v(\omega)d\omega + \int_0^t h_6(\omega)u(\omega)v(\omega)d\omega, \tag{2.2}$$

and $1 > k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega$, then

$$\left.\begin{aligned} u(t) &\le \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \\ v(t) &\le \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \end{aligned}\right\}, \tag{2.3}$$

where

$$f(t) = \max\{h_1(t) + h_4(t), h_2(t) + h_5(t)\}, \qquad g(t) = h_3(t) + h_6(t) \quad \textit{and } k = k_1 + k_2.$$

Proof. First we assume that $k > 0$. Adding inequalities (2.1) and (2.2), we obtain

$$u(t) + v(t) \le \begin{cases} k_1 + k_2 + \int_0^t [h_1(\omega) + h_4(\omega)]u(\omega)d\omega \\ \quad + \int_0^t [h_2(\omega) + h_5(\omega)]v(\omega)d\omega, \\ \quad + \int_0^t [h_3(\omega) + h_6(\omega)]u(\omega)v(\omega)d\omega \end{cases}$$

$$\le \begin{cases} k + \int_0^t f(\omega)u(\omega)d\omega \\ \quad + \int_0^t f(\omega)v(\omega)d\omega \\ + \int_0^t g(\omega)u(\omega)v(\omega)d\omega \end{cases}$$

$$= \begin{cases} k + \int_0^t f(\omega)[u(\omega) + v(\omega)]d\omega \\ \quad + \int_0^t g(\omega)u(\omega)v(\omega)d\omega., \end{cases} \tag{2.4}$$

Substitute $z(t) = u(t) + v(t)$ in (2.4), we get

$$z(t) \leq k + \int_0^t f(\omega)z(\omega)d\omega + \int_0^t g(\omega)z^2(\omega)d\omega. \tag{2.5}$$

Let's denote $w(t)$ by

$$w(t) = k + \int_0^t f(\omega)z(\omega)d\omega + \int_0^t g(\omega)z^2(\omega)d\omega.$$

Therefore,

$$z(t) \leq w(t), \quad w(0) = k \tag{2.6}$$

and

$$w'(t) = f(t)z(t) + g(t)z^2(t) \leq f(t)w(t) + g(t)w^2(t). \tag{2.7}$$

If $w(t) = \frac{1}{y(t)}$, then $w(0) = \frac{1}{y(0)} = k$ and $w'(t) = -\frac{1}{y^2(t)}y'(t)$.

Using above equations in (2.7), we get

$$y'(t) + f(t)y(t) \geq -g(t), \tag{2.8}$$

$$\left[y(t)\exp\left(\int_0^t f(\sigma)d\sigma\right)\right] \geq -g(t)\exp\left(\int_0^t f(\sigma)d\sigma\right). \tag{2.9}$$

From (2.9), we obtain

$$y(t)\exp\left(\int_0^t f(\sigma)d\sigma\right) \geq \frac{1}{k} - \int_0^t g(\omega)\exp\left(\int_0^s f(\sigma)d\sigma\right)d\omega, \tag{2.10}$$

$$y(t)\exp\left(\int_0^t f(\sigma)d\sigma\right) \geq \frac{1 - k\int_0^t g(\omega)\exp\left(\int_0^s f(\sigma)d\sigma\right)d\omega}{k}, \tag{2.11}$$

$$y(t) \geq \frac{1 - k\int_0^t g(\omega)\exp\left(\int_0^s f(\sigma)d\sigma\right)d\omega}{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}, \tag{2.12}$$

$$w(t) \leq \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^s f(\sigma)d\sigma\right)d\omega}. \tag{2.13}$$

From (2.6) and (2.13), we get

$$z(t) \leq \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega}. \tag{2.14}$$

As u, v are nonnegative and $u + v = z$ from inequality (2.14), we get (2.3). In case $k = 0$, we perform the above steps with $k = \varepsilon > 0$ in place of k, where ε is arbitrary constant and hence by limit as $\varepsilon \to 0$ we have (2.3).

Theorem 2.2 *Suppose* $u,, h_i(i = 1\ldots,4)$ *non-negative continuous functions on* $\mathbb{R}_+$ *and* k_1, k_2 *non-negative constants. If*

$$u(t) \le k_1 + \int_0^t h_1(\omega)v(\omega)d\omega + \int_0^t h_2(\omega)u^2(\omega)d\omega \tag{2.15}$$

$$v(t) \le k_2 + \int_0^t h_3(\omega)u(\omega)d\omega + \int_0^t h_4(\omega)v^2(\omega)d\omega, \tag{2.16}$$

for $t \in \mathbb{R}_+$ and $1 > k\int_0^t g(\omega)\exp\left(\int_0^s f(\sigma)d\sigma\right)d\omega$, then

$$\left.\begin{aligned} u(t) &\le \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \\ v(t) &\le \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \end{aligned}\right\}, \tag{2.17}$$

for $t \in \mathbb{R}_+$, where

$$f(t) = \max\{h_1(t), h_3(t)\}, \qquad g(t) = \max\{h_2(t), h_4(t)\}\ and\ k = k_1 + k_2. \tag{2.18}$$

Proof. First we assume that $k > 0$. Adding inequalities (2.15) and (2.16) and making use of (2.18), we obtain

$$u(t) + v(t) \le k + \int_0^t f(\omega)[u(\omega) + v(\omega)]d\omega + \int_0^t g(\omega)[u(\omega) + v(\omega)]^2 d\omega. \tag{2.19}$$

Substitute $z(t) = u(t) + v(t)$ in (2.19), we get

$$z(t) \le k + \int_0^t f(\omega)z(\omega)d\omega + \int_0^t g(\omega)z^2(\omega)d\omega. \tag{2.20}$$

The proof obtained has close resemblance with the proof of the Theorem 2.1 therefore, we discard further steps of the proof.

Theorem 2.3 *Assume that* $u, v, h_i(i = 1\ldots,4)$ *nonnegative continuous functions on* $\mathbb{R}_+$ *and* k_1, k_2 *be nonnegative constants. If*

$$u(t) \le k_1 + \int_0^t h_1(\omega)v^2(\omega)d\omega + \int_0^t h_2(\omega)u(\omega)v(\omega)d\omega \tag{2.21}$$

$$v(t) \le k_2 + \int_0^t h_3(\omega)u^2(\omega)d\omega + \int_0^t h_4(\omega)u(\omega)v(\omega)d\omega, \tag{2.22}$$

for $t \in \mathbb{R}_+$, and $1 > k\int_0^t [f(\omega) + g(\omega)]d\omega$, then

$$u(t) \leq \frac{k}{1 - k\int_0^t [f(\omega) + g(\omega)]d\omega} \text{and} v(t) \leq \frac{k}{1 - k\int_0^t [f(\omega) + g(\omega)]d\omega}, \tag{2.23}$$

for $t \in \mathbb{R}_+$, where $f(t)$, $g(t)$ and k are defined as same in (2.18).

Proof. From inequalities (2.21) and (2.22) and making use of (2.18), we obtain

$$u(t) + v(t) \leq k + \int_0^t f(\omega)[u^2(\omega) + v^2(\omega)]d\omega + \int_0^t g(\omega)u(\omega)v(\omega)d\omega. \tag{2.24}$$

Substitute $z(t) = u(t) + v(t)$ in (2.24), we obtain

$$z(t) \leq k + \int_0^t f(\omega)z^2(\omega)d\omega + \int_0^t g(\omega)z^2(\omega)d\omega, \tag{2.25}$$

$$z(t) \leq k + \int_0^t [f(\omega) + g(\omega)]z^2(\omega)d\omega. \tag{2.26}$$

The proof obtained has close resemblance with the proof of the Theorem 2.1 therefore, we discard further steps of the proof.

Theorem 2.4 *Assume that $u, v, h_i (i = 1...,4)$ non-negative continuous functions on $\mathbb{R}_+$ and k_1, k_2, μ be nonnegative constants. If*

$$u^p(t) \leq k_1 + \int_0^t h_1(\omega)u^p(\omega)d\omega + \int_0^t h_2(\omega)\bar{v}^p(\omega)d\omega \tag{2.27}$$

$$v(t) \leq k_2 + \int_0^t h_3(\omega)\bar{u}^p(\omega)d\omega + \int_0^t h_4(\omega)v(\omega)d\omega, \tag{2.28}$$

for $t \in \mathbb{R}_+$, where $\bar{u}(t) = \exp(-\mu t)u(t)$ and $\bar{v}(t) = \exp(\mu t)v(t)$, then

$$u^p(t) \leq k\exp\left(\mu pt + \int_0^t f(\omega)d\omega\right) \text{ and } v(t) \leq k\exp\left(\int_0^t f(\omega)d\omega\right), \tag{2.29}$$

where

$$f(t) = \max\{h_1(t) + h_3(t), h_2(t) + h_4(t)\}, \quad \text{and } k = k_1 + k_2. \tag{2.30}$$

Proof. Multiply inequality (2.27) by $\exp(-p\mu t)$, we obtain

$$\bar{u}^p(t) \leq k_1 + \int_0^t h_1(\omega)\bar{u}^p(\omega)d\omega + \int_0^t h_2(\omega)v(\omega)d\omega. \tag{2.31}$$

From (2.28) and (2.31), we obtain

$$\bar{u}^p(t) + v(t) \leq k_1 + k_2 + \int_0^t [h_1(\omega) + h_3(\omega)]\bar{u}^p(\omega)d\omega + \int_0^t [h_2(\omega) + h_4(\omega)v(\omega)]d\omega. \tag{2.32}$$

Using (2.30) in (2.32), we get

$$\bar{u}^p(t) + v(t) \le k + \int_0^t f(\omega)[\bar{u}^p(\omega) + v(\omega)]d\omega. \tag{2.33}$$

An application of Gronwall-Bellman's inequality to (2.33), we get

$$\bar{u}^p(t) + v(t) \le k\exp\left(\int_0^t f(\omega)d\omega\right). \tag{2.34}$$

Since $\bar{u}^p(t)$ and $v(t)$ are nonnegative,

$$\bar{u}^p(t) \le k\exp\left(\int_0^t f(\omega)d\omega\right) \text{and} v(t) \le k\exp\left(\int_0^t f(\omega)d\omega\right). \tag{2.35}$$

Substituting the value of $\bar{u}$, we obtain (2.29). This completes the proof.

3. Integro-differenatial Inequalities

In this section, we establish integro-differential inequalities.

Theorem 3.1 *Suppose* $u, v, u', v' h_i (i = 1 \ldots, 4)$ *nonnegative continuous functions on* $\mathbb{R}_+$ *and* k_1, k_2 *nonnegative constants. If*

$$u(t) \le k_1 + \int_0^t h_1(\omega)v'(\omega)d\omega + \int_0^t h_2(\omega)u(\omega)d\omega \tag{3.1}$$

$$v(t) \le k_2 + \int_0^t h_3(\omega)u'(\omega)d\omega + \int_0^t h_4(\omega)v(\omega)d\omega \tag{3.2}$$

for $t \in \mathbb{R}_+$ and $1 > f(t)$, then

$$\left.\begin{aligned} u(t) &\le k\exp\left(\int_0^t \frac{g(\omega)}{1 - f(\omega)}d\omega\right) \\ v(t) &\le k\exp\left(\int_0^t \frac{g(\omega)}{1 - f(\omega)}d\omega\right) \end{aligned}\right\}, \tag{3.3}$$

for $t \in \mathbb{R}_+$, where $f(t), g(t)$ and k are defined as same in (2.18).

Proof. First we assume that $k > 0$. Adding inequalities (3.1) and (3.2) and making use of (2.18), we obtain

$$u(t) + v(t) \le k + \int_0^t g(\omega)[u(\omega) + v(\omega)]d\omega + \int_0^t h_1(\omega)v'(\omega) + h_3(\omega)u'(\omega)]d\omega. \tag{3.4}$$

Substitute $z(t) = \int_0^t [h_1(\omega)u'(\omega) + h_3(\omega)v'(\omega)]d\omega$ in (3.4), we get

$$u(t) + v(t) \le k + z(t) + \int_0^t g(\omega)[u(\omega) + v(\omega)]d\omega. \tag{3.5}$$

$$u'(t)+v'(t)\leq z'(t)+g(t)[u(t)+v(t)]$$

$$\leq f(t)[u'(t)+v'(t)]+g(t)[u(t)+v(t)],$$

$$\frac{u'(t)+v'(t)}{u(t)+v(t)}\leq\frac{g(t)}{1-f(t)}. \tag{3.6}$$

Now (3.6) yields

$$u(t)+v(t)\leq k\exp\left(\int_0^t\frac{g(\omega)}{1-f(\omega)}d\omega\right). \tag{3.7}$$

Since $u(t), v(t)$ are nonnegative from inequality (3.7), we get (3.3).

In case $k=0$, we perform the above steps with $k=\varepsilon>0$ in place of k, where ε is arbitrary constant and hence by limit as $\varepsilon\to 0$ to obtain $u(t)=v(t)=0$. This completes the proof.

Theorem 3.2 *Suppose* $u, v, u', v', h_i(i=1\ldots,4)k_1, k_2$ *as same as defined in Theorem 3.1. If*

$$u(t)\leq k_1+\int_0^t h_1(\omega)\sqrt{u(\omega)+v(\omega)}(\omega)d\omega+\int_0^t h_2(\omega)v'(\omega)d\omega \tag{3.8}$$

$$v(t)\leq k_2+\int_0^t h_3(\omega)\sqrt{u(\omega)+v(\omega)}d\omega+\int_0^t h_4(\omega)u'(\omega)d\omega, \tag{3.9}$$

for $t\in\mathbb{R}_+$ and $1>g(t)$, then

$$\left.\begin{aligned} u(t) &\leq\left(\sqrt{k}+\frac{1}{2}\int_0^t\frac{f(\omega)}{1-g(\omega)}d\omega\right)^2\\ v(t) &\leq\left(\sqrt{k}+\frac{1}{2}\int_0^t\frac{f(\omega)}{1-g(\omega)}d\omega\right)^2\end{aligned}\right\}, \tag{3.10}$$

for $t\in\mathbb{R}_+$, where $f(x)=h_1(x)+h_3(x)$ and $g(x)$ and k have been elaborated in (2.18).

Proof. Adding inequalities (3.8) and (3.9) and making use of (2.18), we obtain

$$u(t)+v(t)\leq k+\int_0^t f(\omega)[\sqrt{u(\omega)+v(\omega)}]d\omega+\int_0^t g(\omega)[u'(\omega)+v'(\omega)]d\omega. \tag{3.11}$$

Differentiating (3.11), we have

$$u'(t)+v'(t)\leq f(t)\left[\sqrt{u(t)+v(t)}\right]d\omega+g(t)[u'(t)+v'(t)], \tag{3.12}$$

$$\frac{u'(t)+v'(t)}{\sqrt{u(t)+v(t)}}\leq\frac{f(t)}{1-g(t)}. \tag{3.13}$$

The inequality (3.6) yields

$$u(t)+v(t)\le\left(\sqrt{k}+\frac{1}{2}\int_0^t\frac{f(\omega)}{1-g(\omega)}d\omega\right)^2. \tag{3.14}$$

Since $u(t), v(t)$ are nonnegative from inequality (3.14), we get (3.10).

The proof is complete.

Theorem 3.3 *Let* $u, v, u', v' h_i (i = 1 \ldots, 4) k_1, k_2$ *be as same as defined in Theorem 3.1. If*

$$\sqrt{u(t)}\le k_1+\int_0^t h_1(\omega)\sqrt{u(\omega)+v(\omega)}(\omega)d\omega+\int_0^t h_2(\omega)v'(\omega)d\omega \tag{3.15}$$

$$\sqrt{v(t)}\le k_2+\int_0^t h_3(\omega)\sqrt{u(\omega)+v(\omega)}d\omega+\int_0^t h_4(\omega)u'(\omega)d\omega, \tag{3.16}$$

for $t\in\mathbb{R}_+$ and $1 > g(t)$, then

$$\left.\begin{aligned} u(t) &\le (k_1^2+k_2^2)\exp\left(2\int_0^t f(\omega)d\omega\right)\\ v(t) &\le (k_1^2+k_2^2)\exp\left(2\int_0^t f(\omega)d\omega\right)\end{aligned}\right\}, \tag{3.17}$$

for $t\in\mathbb{R}_+$, where $f(x) = h_1(x) + h_3(x)$ and $g(x) = \max\{h_2(x), h_4(x)\}$.

Proof. Adding inequalities (3.8) and (3.9) and making use of (2.18), we obtain

$$\sqrt{u(t)}+\sqrt{v(t)}\le k_1+k_2+\int_0^t f(\omega)\left[\sqrt{u(\omega)+v(\omega)}\right]d\omega$$
$$+\int_0^t g(\omega)[u'(\omega)+v'(\omega)]d\omega, \tag{3.18}$$

$$\sqrt{u(\omega)+v(\omega)}\le k_1+k_2+\int_0^t f(\omega)\left[\sqrt{u(\omega)+v(\omega)}\right]d\omega$$
$$+\int_0^t g(\omega)[u'(\omega)+v'(\omega)]d\omega. \tag{3.19}$$

Differentiating (3.19), we get

$$\frac{u'(t)+v'(t)}{2\sqrt{u(t)+v(t)}}\le f(t)\left[\sqrt{u(t)+v(t)}\right]+g(t)[u'(t)+v'(t)], \tag{3.20}$$

$$\frac{u'(t)+v'(t)}{u(t)+v(t)}\le 2f(t), \tag{3.21}$$

(3.21) yields,

$$u(t) + v(t) \leq (k_1^2 + k_2^2)\exp\left(2\int_0^t f(\omega)d\omega\right). \tag{3.22}$$

Since u, v are nonnegative from inequality (3.14), we get (3.10).

This completes the proof.

4. Applications

Consider the following system:

$$\left.\begin{aligned} \frac{dx}{dt} &= F(t, x(t), y(t)), \\ \frac{dy}{dt} &= G(t, x(t), y(t)) \end{aligned}\right\}, \tag{4.1}$$

with initial conditions

$$x(0) = k_1, y(0) = k_2, \tag{4.2}$$

where $t \in \mathbb{R}_+, x, y \in C[\mathbb{R}_+, \mathbb{R}], F, G \in C[\Delta, \mathbb{R}], \Delta = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and k_1, k_2 are constants.

Corollary 4.1 *Consider the system (4.1) with initial conditions (4.2). If the functions F and G satisfy the following conditions:*

$$\left.\begin{aligned} |F(t, x(t), y(t))| &\leq h_1|x| + h_2|y| + h_3|x||y|, \\ |G(t, x(t), y(t))| &\leq h_4|x| + h_5|y| + h_6|x||y| \end{aligned}\right\} \tag{4.3}$$

and if there exists a real number $b \in \mathbb{R}_+$ such that $1 > k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega$ for all $t \in [0, b]$, then solution (x, y) of (4.1) is bounded on $[0, b]$.

where h_i are same as defined in the theorem 2.1.

Proof. From (4.1), we obtain

$$\left.\begin{aligned} x(t) &= k_1 + \int_0^t F(s, x(\omega), y(\omega))d\omega, \\ y(t) &= k_2 + \int_0^t G(\omega, x(\omega), y(\omega))d\omega \end{aligned}\right\}, \tag{4.4}$$

Hence,

$$\left.\begin{aligned} |x(t)| &\leq |k_1| + \int_0^t |F(s, x(\omega), y(\omega))|d\omega, \\ |y(t)| &\leq |k_2| + \int_0^t |G(\omega, x(\omega), y(\omega))|d\omega \end{aligned}\right\}, \tag{4.5}$$

Using conditions (4.3) in (4.5), we get

$$\left.\begin{aligned} |x(t)| &\leq |k_1| + \int_0^t h_1(\omega)|x|d\omega + \int_0^t h_2(\omega)|y|d\omega + \int_0^t h_3(\omega)|x||y|d\omega, \\ |y(t)| &\leq |k_2| + \int_0^t h_4(\omega)|x|d\omega + \int_0^t h_5(\omega)|y|d\omega + \int_0^t h_6(\omega)|x||y|d\omega \end{aligned}\right\}. \tag{4.6}$$

Applying the Theorem 2.1, we obtain

$$\left.\begin{aligned} |x(t)| &\leq \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \\ |y(t)| &\leq \frac{k\exp\left(\int_0^t f(\sigma)d\sigma\right)}{1 - k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega} \end{aligned}\right\},$$

for all $t \in [0, b]$, where f, g, k are defined as same in the theorem 2.1. Thus, solution (x, y) of (4.1) are bounded on $[0, b]$.

Example 4.1 *Consider the following simultaneous integral equations*

$$x(t) \leq \frac{1}{2} + \int_0^t e^{-\omega}x(\omega)d\omega + \int_0^t y(\omega)d\omega + \int_0^t \omega x(\omega)y(\omega)d\omega \tag{4.7}$$

$$y(t) \leq \frac{1}{2} + \int_0^t 2x(\omega)d\omega + \int_0^t e^{-2\omega}y(\omega)d\omega + \int_0^t x(\omega)y(\omega)d\omega. \tag{4.8}$$

Then the condition $1 > k\int_0^t g(\omega)\exp\left(\int_0^\omega f(\sigma)d\sigma\right)d\omega$ is hold for all $t \in \left[0, \frac{1}{3}\right]$, where f, g, k are as same as defined in the theorem 2.1. An application of the theorem 2.1, we obtain $x(t), y(t) < \frac{9e^{3t}}{7}$. for all $t \in \left[0, \frac{1}{3}\right]$. Thus, we have obtained known explicit bound on solution of simultaneous integral equation 4.7 and 4.8.

Example 4.2 *Consider the following simultaneous integro-differential equation:*

$$x(t) \leq 2 + \int_0^t \frac{e^{-\omega}y'(\omega)}{2}d\omega + \int_0^t x(\omega)d\omega \tag{4.9}$$

$$y(t) \leq 3 + \int_0^t \frac{e^{-2\omega}}{5}x'(\omega)d\omega + \int_0^t 2y(\omega)d\omega, \tag{4.10}$$

for $t \in \mathbb{R}_+$. Then the condition $1 > f(t)$ is hold for $t \in \mathbb{R}_+$. Applying the theorem 3.1, we obtain

$$\left.\begin{aligned} x(t) &\leq k\exp\left(\int_0^t \frac{g(\omega)}{1 - f(\omega)}d\omega\right) = 5e^{4t} \\ y(t) &\leq k\exp\left(\int_0^t \frac{g(\omega)}{1 - f(\omega)}d\omega\right)) = 5e^{4t} \end{aligned}\right\}, \tag{4.11}$$

for $t \in \mathbb{R}_+$, where $f(t), g(t)$ and k are defined as same in (2.18). Here, we have obtained known explicit estimate on unknown functions x and y.

Mathematical model for Gonorrhea disease

The following model is a testimony of Gonorrhea disease:

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1x + b_1(c_1 - x)y,\\ \frac{dy}{dt} &= a_2y + b_2(c_2 - y)x.\end{aligned}\right\}, \tag{4.12}$$

Corollary 4.2 *If the functions* F *and* G *satisfies the following conditions:*

$$\left.\begin{aligned}|F(t,x,y)| &\leq a_1x + b_1(c_1 - x)y\\ |G(t,x,y)| &\leq a_2y + b_2(c_2 - y)x\end{aligned}\right\},$$

then a solution of the system (4.12) is bounded, where a_i, b_i and c_i are same as defined in (4.12).

The proof of the Corollary 4.2 is similar to the Corollary 4.1, and hence, we omit it here.

Example 4.3 *Consider the following simultaneous differential equation*

$$\left.\begin{aligned}\frac{dx}{dt} &= x\\ \frac{dy}{dt} &= y\end{aligned}\right\}, \tag{4.13}$$

with initial conditions $x(0) = 2, y(0) = 3$,where $t \in [0,22], x, y \in C[[0,22], \mathbb{R}]$.

Then from (4.13), we obtain

$$\left.\begin{aligned}|x(t)| &\leq 2 + x(t),\\ |y(t)| &\leq 3 + y(t)\end{aligned}\right\}.$$

Making use of theorem 2.1, we get

$$\left.\begin{aligned}x(t) &\leq 5\exp\left(\int_0^t d\sigma\right)\\ v(t) &\leq 5\exp\left(\int_0^t d\sigma\right)\end{aligned}\right\}. \tag{4.14}$$

If we choose $t \in [0,22]$, then we have $x(t), y(t) \leq 5e^{22}$. Thus, solution of (4.13) is bounded on $t \in [0,22]$.

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