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THEORETICAL ASSEPECTS OF THE LAWS OF GENERAL RELATIVITY

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ABSTRACT

This paper shows the theoretical consideration of the laws of general relativity using Einstein equations, linking the curvature of space time. It is found that the g is a solution of the Einstein equations and f is a diffeomorphism of the manifold onto itself, then the pullback f^*g is also a solution.

INDEX-TERM: Relativity, Einstein equations & Space time.

INTRODUCTION

The laws of General Relativity are the Einstein equations, linking the curvature of space time to its matter content.[1-2]

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu} \tag{1}$$

(Rationalized units $4\pi G = 1$.) Here $T_{\mu\nu}$ is the energy-momentum tensor of matter, $G_{\mu\nu}$ the Einstein tensor , $R_{\mu\nu}$ the Ricci tensor[3-4] and R the scalar curvature of the metric $g_{\mu\nu}$.

According to original Bianchi identity

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} := \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0 \ , \tag{2}$$

one obtains

$$\nabla^\nu G_{\mu\nu} = 0 \ , \tag{3}$$

the twice contracted Bianchi identity . This identity (3) implies

$$\nabla^\nu T_{\mu\nu} = 0 \ , \tag{4}$$

the equations of motion of matter.

The Einstein vacuum equations are

$$T_{\mu\nu} = 0 \tag{5}$$

Then the equations $G_{\mu\nu} = 0$ are equivalent to

$$R_{\mu\nu} = 0 \tag{6}$$

The curvature tensor[5] in arbitrary local coordinates is as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) \tag{7}$$

$$R^{\alpha}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\lambda} \tag{8}$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{9}$$

P.P. denotes the principal part, i.e. part containing the highest (i.e. 2nd) derivatives.

$$\text{P.P.}\{R_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} \{\partial_\mu \partial_\alpha g_{\beta\nu} + \partial_\nu \partial_\alpha g_{\beta\mu} - \partial_\mu \partial_\nu g_{\alpha\beta} - \partial_\alpha \partial_\beta g_{\mu\nu}\} \qquad 10)$$

The Symbol. we substitute in the principal part

$$\partial_\mu \partial_\nu g_{\alpha\beta} \quad \text{by} \quad \xi_\mu \xi_\nu \dot{g}_{\alpha\beta}$$

where ξ_μ are the components of a covector and $g_{\alpha\beta}$ the components of a possible variation of g.

ANALYSIS

We first obtain the symbol σ_ξ at a point $p \in M$ and a covector $\xi \in T^*_pM$ for a given metric g

$$(\sigma_\xi \cdot \dot{g})_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\xi_\mu \xi_\alpha \dot{g}_{\beta\nu} + \xi_\nu \xi_\alpha \dot{g}_{\beta\mu} - \xi_\mu \xi_\nu \dot{g}_{\alpha\beta} - \xi_\alpha \xi_\beta \dot{g}_{\mu\nu})$$

Here we consider,

$$\begin{aligned} (i_\xi \dot{g})_\nu &= g^{\alpha\beta} \xi_\alpha \dot{g}_{\beta\nu} , \\ (\xi, \xi) &= g^{\alpha\beta} \xi_\alpha \xi_\beta , \\ (\xi \otimes \zeta)_{\mu\nu} &= \xi_\mu \zeta_\nu , \\ g^{\alpha\beta} \dot{g}_{\alpha\beta} &= tr\dot{g} . \end{aligned}$$

Then we can write

$$(\sigma_\xi \cdot \dot{g}) \;=\; \frac{1}{2}\{\xi \otimes i_\xi \dot{g} + i_\xi \dot{g} \otimes \xi - tr\dot{g}\xi \otimes \xi - (\xi,\xi)\dot{g}\}$$

The general situation is as follows. We have a Lagrangian L = L(x, q, v). Here,

x. independent variables x^μ, μ = 1, ... , n

q. dependent variables q^a, a = 1, ... , m

v. 1st derivatives of dependent variables v^a_μ , n × m matrices .

Solution of the Euler-Lagrange equations. For a solution ua of the Euler-Lagrange equations we substitute.

$$\begin{aligned} q^a &= u^a(x) \\ v^a_\mu &= \frac{\partial u^a}{\partial x^\mu}(x) \end{aligned}$$

and require.

$$\frac{\partial}{\partial x^\mu}\{\frac{\partial L}{\partial v^a_\mu}(x, u(x), \partial u(x))\} - \frac{\partial L}{\partial q^a}(x, u(x), \partial u(x)) \;=\; 0 \qquad (11)$$

Defining

$$\begin{aligned} p^\mu_a &= \frac{\partial L}{\partial v^a_\mu} \;, \\ f_a &= \frac{\partial L}{\partial q^a} \;, \end{aligned}$$

the Euler-Lagrange equations become.

$$\frac{\partial p^\mu_a}{\partial x^\mu} \;=\; f_a$$

The q, v, p, f are analogous to position, velocity, momentum and force, respectively, in classical mechanics.
P.P. of the Euler-Lagrange equations.

$$\begin{aligned} & h^{\mu\nu}_{ab}\frac{\partial^2 u^b}{\partial x^\mu \partial x^\nu}(x, u(x), \partial u(x)) \;, \\ & h^{\mu\nu}_{ab} \;=\; \frac{\partial^2 L}{\partial v^a_\mu \partial v^b_\nu}\;(x, q, v) \;. \end{aligned}$$

Linearized equations. The P.P. of the linearized equations is $h_{ab}^{\mu\nu}(x, u(x), \partial u(x)) \frac{\partial^2 u^b}{\partial x^\mu \partial x^\nu}$. The u^a are the variations of the unknown functions. We consider highly oscillatory solutions

$$\dot{u}^a = \dot{w}^a e^{i\Phi} \qquad (12)$$

Write $\frac{\Phi}{\varepsilon}$ in place of Φ. Substitute in the actual linearized equations and keep only the leading terms as $\varepsilon \to 0$. This will give

$$h_{ab}^{\mu\nu}(x, u(x), \partial u(x)) \; \dot{w}^b \; \frac{\partial \Phi}{\partial x^\mu} \; \frac{\partial \Phi}{\partial x^\nu} \qquad (13)$$

This is the symbol $\sigma_\xi \omega$, where $\xi_\mu = \frac{\partial \Phi}{\partial x^\mu}$. Thus, in general the symbol of the Euler-Lagrange equations is

$$(\sigma_\xi \cdot \dot{u})^a = h_{ab}^{\mu\nu} \xi_\mu \xi_\nu \dot{u}^b \qquad (14)$$

$$= \chi_{ab}(\xi) \dot{u}^b \qquad (15)$$

where $\chi_{ab}(\xi) = h_{ab}^{\mu\nu} \xi_\mu \xi_\nu$ is an m×m matrix whose entries are homogeneous quadratic polynomials in ξ.

Let M be a n-dimensional manifold. Then the characteristic subset $C^*_x \subset T^*_x M$ is defined by

$$\begin{aligned} C^*_x &= \{\xi \neq 0 \; \in \; T^*_x M : \; null\ space\ of\ \sigma_\xi \neq 0\} \\ &= \{\xi \neq 0 \; \in \; T^*_x M : \; det\chi(\xi) = 0\}. \end{aligned}$$

If $\xi \in C^*_x$, then we have non-trivial null space of σ_ξ.

Simplest Example with a non-empty characteristic. linear wave equation.

$$u\ . = g^{\mu\nu} \nabla_\mu (\partial_\nu u) = 0$$

The symbol is $\sigma_\xi\, u = (g^{\mu\nu} \xi_\mu\, \xi_\nu) u$. And we find the characteristic to be $C^*_x = \{\xi \neq 0 \in T^*_x M\ .\ (\xi\ ,\ \xi) = g^{\mu\nu} = \xi_\mu\, \xi_\nu = 0\}$, which means C^*_x is a null cone in $T^*_x M$.

Now return to the symbol for the Einstein equations. Set

$$\dot{g} = \zeta \otimes \xi + \xi \otimes \zeta \qquad (16)$$

for an arbitrary covector $\zeta \in T^*_x M$. Then

$$i_\xi \dot{g} = \underbrace{(\zeta,\xi)}_{=g^{\mu\nu}\zeta_\mu\xi_\nu} \xi + (\xi,\xi)\,\zeta \quad (17)$$

and

$$tr\dot{g} = 2(\zeta,\xi) \quad (18)$$

We see that

$$\sigma_\xi \cdot \dot{g} = 0 \quad (19)$$

CONCLUSION

This paper summarized the null space which is non-trivial for every covector ζ. This is due to the fact that the equations are generally covariant. I.e. If X is a (complete) vector field on M then X generates a 1-parameter group $\{f_t\}$ of diffeomorphisms of M and

$$\mathcal{L}_X g = \frac{d}{dt} f_t^* g|_{t=0}$$

the Lie derivative with respect to X of g, is a solution of the linearized equations.

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