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ORIGINAL ARTICLE

THE DERIVED PICARD GROUP IS A LOCALLY ALGEBRAIC GROUP

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Abstract.-

Let A be a finite dimensional algebra over an algebraically closed field K. The derived Picard group $DPic_k(A)$ is the group of two-sided tilting complexes over A modulo isomorphism. We prove that $DPic_k(A)$ is a locally algebraic group, and its identity component is $Out^0_K(A)$. If B is a derived Morita equivalent algebra then $DPic_K(A) \cong DPic_K(B)$ as locally algebraic groups.

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Key words: Picard group, locally algebraic group.

Let A and B be associative algebras with 1 over a field K. We denote by D^b (Mod A) the bounded derived category of left A-modules. Let B° be the op- posite algebra, so an $A \times_K B°$-module is a K-central A-B-bimodule. A two-sided tilting complex over (A, B) is a complex $T \in D^b$ (Mod $A \times_k B°$) such that there exists a complex $T^v \in D^b$ (Mod $B \times A°$) and isomorphisms of the derived tensor products $T \times {}^L_B T^V \cong A$ and $T^V \times {}^L_A T \cong B$. Two-sided tilting complexes were introduced by Rickard in [Rd]

When B = A we write $A^e := A \times_K A°$. The set

$$DPicK(A) = \frac{\{two-sided\ titling\ complexes\ T \in D^b(Mod A^e)\}}{isomorphism}$$

is the derived Picard group of A (relative to K). The identity element is the class of A, the multiplication is $(T_1, T_2) \to T_1 \times {}^L_A T_2$, and the inverse is $T \to T^V = RHom_A(T, A)$.

Denote by $Out_K(A)$ the group of outer K-algebra automorphism of A, and by $Pic_K(A)$ the Picard group of A (the group of invertible bimodules modulo isomor- phism). Then there are inclusions

$Out_K (A) \subset Pic_K(A) \subset DPic_K(A)$.

The first inclusion sends the automorphism σ to the invertible bimodulc A^{σ} where the right action is twisted byσ. The second inclusion corresponds to the full embedding Mod $A^{e} \subset D^{b}$ (Mod A^{e}). See [Y_{e}] for details.

To simplify notation we use the same symbol to denote an automorphism $\sigma \in \mathrm{Aut}_{K}$ (A) and its class in Out_{k}(A). Likewise for a two-sided tilting complex T and Its class in DPic_{K}(A). The precise meaning is always clear from the context. Now assume K is algebraically closed and A is a finite dimensional K-algebra. Then the group $\mathrm{Aut}_{K}(A) = \mathrm{AutAIg}_{K}(A)$ of K-algebra automorphisms is a linear Algebraic group, being a closed subgroup of GL(A) = $\mathrm{Aut}_{\mathrm{ModK}}$(A)- This induces a Structure of linear algebraic group on the quotient Out_{K}(A). Denote by Out $^{0}_{K}(A)$the identity component.

Examples calculated in [MY] indicated that the whole group DPic_{K}(A) should carry a geometric structure (cf. Example 3 below). This is our first main result .

Theorem 2.

A result of Brauer says that the group Out $^{0}_{K}(A)$ is a Morita invariant of A: if A and B are Morita equivalent K-algebras then Out $^{0}_{K}(A) \cong Out^{0}_{K}(B)$. In [HS] and [Ro] this is extended to derived Morita equivalence. Our Theorem 4 extends these results further. We shall need the following variant of the result of Huisgen-Zimmermann, Saorin and Rouquier.

THEOREM 1. Let A and B be finite dimensional K-algebras. Suppose $T \in D^{b}$ ($\mathrm{ModA} \times_{K} B°$) is a two-sided titling complex over (A, B), with inverse $T^{v} \in D^{b}(\mathrm{ModB} \times_{K} A°)$. Then for any element $\delta \in Out^{0}_{K}$ (A) the two-sided tilting complex $\emptyset^{0}_{T}(\delta) \coloneqq T^{v} \times {}^{L}_{A} A^{\sigma} \times {}^{L}_{A} T \in DPic_{k}$(B) Is in $Out^{0}_{K}(B)$. The group homomorphism

$$\emptyset^{0}_{T}: Out^{0}_{K}(A) \rightarrow Out^{0}_{K}(B)$$

is an isomorphism of algebraic groups.

Proof. According to [HS, Theorem 17] or [Ro, Theoreme 4.2] there is an iso- morphism of algebraic groups $\emptyset^{0}$: Out^{0}_{K} (A) $\rightarrow Out^{0}_{K}$(B) induced by T. Letting

$\tau := \emptyset^{0}(\sigma) \in Out^{0}_{K}(B)$ one has
$T \times_{B} B^{i} \cong A^{\sigma} \times_{A} T \; in \; D(Mod \times_{K} D^{0})$.

Applying $T^{V} \times^{L}_{A}$ to this isomorphism we see that $B^{i} \cong \emptyset^{0}_{T}(\sigma) in \; D(Mod \; B^{e}), so$

$i = \emptyset^{0}_{T}(\sigma) \; in \; DPic_{K}(B). We \; conclude \; that \; \emptyset^{0}_{T} = \emptyset^{0}.$

A locally algebraic group over K is a group G, with a normal subgroup G°, such that G° is a connected algebraic group over K, each coset of G° is a variety, and multiplication and inversion are morphisms of varieties. A morphism $\emptyset: G \rightarrow H$ of locally algebraic groups is a group homomorphism such that $\emptyset(G^{0}) \subset H^{0}$ And the restriction $\emptyset^{0}: G^{0} \rightarrow H^{0}$ is a morphism of varieties. We call $\emptyset$ can open immersion if $\emptyset$ is injective and $\emptyset^{0}$ is an isomorphism.

In other words G is the group of rational points G(K) of a reduced group scheme G locally of finite type over K, in the sense of [SGA3, Expose VI_A]. A morphism $\emptyset = G \to H$ corresponds to a morphism $\emptyset: G \to H$ of group schemes over K.
Here is our first main result.

Theorem 2. Let A be a finite dimensional K-algebra. Then the derived Picard group $DPic_K(A)$ is a locally algebraic group over K. The inclusion $Out_K(A) \subset DPic_K(A)$ is an open immersion.

In particular the identity components coincide: $Out_K^0(A) = DPic_K^0(A)$. Proof. Theorem 1 with A = B implies that the subgroup $Out_K^0(A) \subset DPic_K(A)$ is normal, and for any two-sided tilting complex T the conjugation cf $\emptyset_T^0: Out_K^0(A) \to Out_K^0(A)$ is an automorphism of algebraic groups.

Let us now switch to the notation $T_1.T_2$ *and* T^{-1} for the operations in $DPic_K(A)$. Define an algebraic variety structure on each coset C=T. $Out_K^0(A) \subset DPic_K(A)$ using the multiplication map $P \to T.P, P \in Out_K^0(A)$. Since $\emptyset_T^0$ is an automor- phism of algebraic groups, the variety structure is independent of the representative T∈C.

Let us prove that $DPic_K(A)$ is a locally algebraic group. For $P_1, P_2 \in Out_K^0(A)$ and $T_1, T_2 \in DPic_K(A)$, multiplication is the morphism $(T_1.P_1)(T_2.P_2) = (T_1.T_2).(\emptyset_{T_2}^0(P_1).P_2)$.

Similarly for the inverse:

$$(T.P)^{-1} = T^{-1}.\emptyset_T^0(P)^{-1}$$

Example 3. Let $\overrightarrow{\Omega_n}$ be the quiver with two vertices x, y and n arrows $x\underline{\alpha}iy$.
Let A be the path algebra $K\overrightarrow{\Omega_n}$ According to [MY, Theorem 5.3], $Out_K(A) \cong Pic_K(A) \cong PGL_n(K)$ and

$DPic_K(A) \cong Z \times (Z \times PGL_n(K))$.

In the semi-direct product a generator T of Z acts on a matrix $\sigma \in PGL_n(K) by\ \emptyset_T^0(\sigma) = (\sigma^{-1})^t$. This is clearly a morphism of varieties, so $DPic_K(A)$ is indeed a locally algebraic group.

Our second main result relates two algebras. Recall that the algebras A and B are derived Morita equivalent over K if there is a K-linear equivalence of triangulated categories $D^b(ModA) \approx D^b(ModB)$.

Theorem 4. Suppose A and B are two finite dimensional K-algebras, and assume they are derived Morita equivalent over K. Then $DPic_K(A) \cong DPic_K(B)$ as locally algebraic groups.

Proof. It is known that there exist two-sided tilting complexes $T \in D(ModA \times_K B^0)$; choose one. We obtain a group isomorphism

$$\emptyset_T: \begin{cases} DPic_k(A) \to DPic_k(B) \\ S \to T^V \times {}_A^L S \times {}_A^L T \end{cases}.$$

By Theorem 1, $\emptyset_T$ restricts to an isomorphism of algebraic groups $\emptyset_T^0 : Out_K^0(A) \rightarrow Out_K^0(B). So\ \emptyset_T$ is an isomorphism of locally algebraic groups.

We end the paper with a corollary and some remarks. Suppose C is a K-linear triangulated category that's equivalent to a small category. Denote by $Out_K^{tr}(C)$the group of K-linear triangle auto-equivalences of C modulo natural isomorphism.
Let mod A stand for the category of finitely generated A- modules.

Corollary 5. Suppose C is a K-linear triangulated category that is equivalent to D^b (mod a) for some hereditary finite dimensional K.-algebra A. Then Out_K^{tr} (C) is a locally algebraic group.
Proof. Trivially $Out_K^{tr}(C) \cong Out_K^{tr}(D^b(mod\ A))$, and by [MY, Corollary 0.11] we have $Out_K^{tr}(D^b(mod\ A)) \cong DPic_K(A)$.

Example 6. Beilinson [Be] proved that D^b (CohP_K^1)≈ D^b(mod K $\overrightarrow{\Omega_2}$), where Coh$P_K^1$ is the category of coherent sheaves on the projective line, and $\overrightarrow{\Omega_2}$ is the quiver from Example 3. Therefore, $Out_K^{tr}(D^b(CohP_K^1))$is a locally algebraic group.
This should be compared to Remark 7 below; see also [MY, Remark 5.4].

Remark 7. Suppose X is a smooth projective variety over K with ample canonical or anti-canonical bundle. Bondal and Orlov [BO] proved that

$$Out_K^{tr}\left(D^b(CohX)\right) \cong (Aut_k(X) \times Pic(X)) \times Z$$

Here Pic(X) is the group of line bundles. Thus, $Out_K^{tr}(D^b(CohX))$≅G×D, where G is an algebraic group and D is a discrete group, and in particular, this is a locally algebraic group.

Remark 8. In [Or] , Orlov gives an example of an abelian variety over IK such that

$$Out_K^{tr}\left(D^b(CohX)\right) \cong D \times \left(X \times \hat{X}\right)(K)$$

where D is a discrete group (an extension of SL_2(Z) by Z) and $\hat{X}$ is the dual Abelian variety. The group D acts (nontrivially) via Aut_K(X x $\hat{X}$) and hence $Out_K^{tr}(D^b(CohX))$ is a locally algebraic group.

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