Research Papers

Philosophy of Mathematics

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Abstract

The philosophy of Mathematics is the branch of philosophy that studies the philosophical assumptions, foundations, and implications of mathematics. The aim of the philosophy of mathematics is to provide an account of the nature and methodology of mathematics and to understand the place of mathematics in people's lives. The logical and structural nature of mathematics itself makes this study both broad and unique among its philosophical counterparts.

The terms philosophy of mathematics and mathematical philosophy are frequently used as synonyms.[1] The latter, however, may be used to refer to several other areas of study. One refers to a project of formalising a philosophical subject matter, say, aesthetics, ethics, logic, metaphysics, or theology, in a purportedly more exact and rigorous form, as for example the labours of Scholastic theologians, or the systematic aims of Leibniz and Spinoza. Another refers to the working philosophy of an individual practitioner or a like-minded community of practicing mathematicians. Additionally, some understand the term "mathematical philosophy" to be an allusion to the approach taken by Bertrand Russell in his books The Principles of Mathematics and Introduction to Mathematical Philosophy.

History

The origin of mathematics is subject to argument. Whether the birth of mathematics was a random happening or induced by necessity duly contingent of other subjects, say for physics, is still a matter of prolific debates.

Many thinkers have contributed their ideas concerning the nature of mathematics. Today, some philosophers of mathematics aim to give accounts of this form of inquiry and its products as they stand, while others emphasize a role for themselves that goes beyond simple interpretation to critical analysis. There are traditions of mathematical philosophy in both Western philosophy and Eastern philosophy. Western philosophies of mathematics go as far back as Plato, who studied the ontological status of mathematical objects, and Aristotle, who studied logic and issues related to infinity (actual versus potential).

Greek philosophy on mathematics was strongly influenced by their study of geometry. For example, at one time, the Greeks held the opinion that 1 (one) was not a number, but rather a unit of arbitrary length. A number was defined as a

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multitude. Therefore 3, for example, represented a certain multitude of units, and was thus not "truly" a number. At another point, a similar argument was made that 2 was not a number but a fundamental notion of a pair. These views come from the heavily geometric straight-edge-and-compass viewpoint of the Greeks: just as lines drawn in a geometric problem are measured in proportion to the first arbitrarily drawn line, so too are the numbers on a number line measured in proportional to the arbitrary first "number" or "one."

These earlier Greek ideas of numbers were later upended by the discovery of the irrationality of the square root of two. Hippasus, a disciple of Pythagoras, showed that the diagonal of a unit square was incommensurable with its (unit-length) edge: in other words he proved there was no existing (rational) number that accurately depicts the proportion of the diagonal of the unit square to its edge. This caused a significant re-evaluation of Greek philosophy of mathematics. According to legend, fellow Pythagoreans were so traumatized by this discovery that they murdered Hippasus to stop him from spreading his heretical idea. Simon Stevin was one of the first in Europe to challenge Greek ideas in the 16th century. Beginning with Leibniz, the focus shifted strongly to the relationship between mathematics and logic. This perspective dominated the philosophy of mathematics through the time of Frege and of Russell, but was brought into question by developments in the late 19th and early 20th century.

20th century

A perennial issue in the philosophy of mathematics concerns the relationship between logic and mathematics at their joint foundations. While 20th century philosophers continued to ask the questions mentioned at the outset of this article, the philosophy of mathematics in the 20th century was characterized by a predominant interest in formal logic, set theory, and foundational issues.

It is a profound puzzle that on the one hand mathematical truths seem to have a compelling inevitability, but on the other hand the source of their "truthfulness" remains elusive. Investigations into this issue are known as the foundations of mathematics program.

At the start of the 20th century, philosophers of mathematics were already beginning to divide into various schools of thought about all these questions, broadly distinguished by their pictures of mathematical epistemology and ontology. Three schools, formalism, intuitionism, and logicism, emerged at this time, partly in response to the increasingly widespread worry that mathematics as it stood, and analysis in particular, did not live up to the standards of certainty and rigour that had been taken for granted. Each school addressed the issues that came to the fore at that time, either attempting to resolve them or claiming that mathematics is not entitled to its status as our most trusted knowledge.

Surprising and counter-intuitive developments in formal logic and set theory early in the 20th century led to new questions concerning what was traditionally called the foundations of mathematics. As the century unfolded, the initial focus of concern expanded to an open exploration of the fundamental axioms of mathematics, the axiomatic approach having been taken for granted since the time of Euclid around 300 BCE as the natural basis for mathematics. Notions of axiom, proposition and proof, as well as the notion of a proposition being true of a mathematical object (see Assignment (mathematical logic)), were formalized, allowing them to be treated mathematically. The Zermelo-Fraenkel axioms for set theory were formulated which provided a conceptual framework in which much mathematical discourse would be interpreted. In mathematics as in physics, new and unexpected ideas had arisen and significant changes were coming. With Gödel numbering, propositions could be interpreted as referring to themselves or other propositions, enabling inquiry into the consistency of mathematical theories. This reflective critique in which the theory under review "becomes itself the object of a mathematical study" led Hilbert to call such study metamathematics or proof theory.[2]

At the middle of the century, a new mathematical theory was created by Samuel Eilenberg and Saunders Mac Lane, known as category theory, and it became a new contender for the natural language of mathematical thinking (Mac Lane 1998). As the 20th century progressed, however, philosophical opinions diverged as to just how well-founded were the questions about foundations that were raised at its opening. Hilary Putnam summed up one common view of the situation in the last third of the century by saying:

When philosophy discovers something wrong with science, sometimes science has to be changed — Russell's paradox comes to mind, as does Berkeley's attack on the actual infinitesimal — but more often it is philosophy that has to be changed. I

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do not think that the difficulties that philosophy finds with classical mathematics today are genuine difficulties; and I think that the philosophical interpretations of mathematics that we are being offered on every hand are wrong, and that "philosophical interpretation" is just what mathematics doesn't need. (Putnam, 169-170).

Philosophy of mathematics today proceeds along several different lines of inquiry, by philosophers of mathematics, logicians, and mathematicians, and there are many schools of thought on the subject. The schools are addressed separately in the next section, and their assumptions explained.

Contemporary schools of thought

Mathematical realism, like realism in general, holds that mathematical entities exist independently of the human mind. Thus humans do not invent mathematics, but rather discover it, and any other intelligent beings in the universe would presumably do the same. In this point of view, there is really one sort of mathematics that can be discovered: Triangles, for example, are real entities, not the creations of the human mind.

Many working mathematicians have been mathematical realists; they see themselves as discoverers of naturally occurring objects. Examples include and Kurt Gödel. Gödel believed in an objective mathematical reality that could be perceived in a manner analogous to sense perception. Certain principles (e.g., for any two objects, there is a collection of objects consisting of precisely those two objects) could be directly seen to be true, but some conjectures, like the continuum hypothesis, might prove undecidable just on the basis of such principles. Gödel suggested that quasi-empirical methodology could be used to provide sufficient evidence to be able to reasonably assume such a conjecture.

Within realism, there are distinctions depending on what sort of existence one takes mathematical entities to have, and how we know about them.

Platonism

Mathematical Platonism is the form of realism that suggests that mathematical entities are abstract, have no spatiotemporal or causal properties, and are eternal and unchanging. This is often claimed to be the view most people have of numbers. The term Platonism is used because such a view is seen to parallel Plato's Theory of Forms and a "World of Ideas" (Greek: Eidos (ε δος)) described in Plato's Allegory of the cave: the everyday world can only imperfectly approximate an unchanging, ultimate reality. Both Plato's cave and Platonism have meaningful, not just superficial connections, because Plato's ideas were preceded and probably influenced by the hugely popular Pythagoreans of ancient Greece, who believed that the world was, quite literally, generated by numbers.

The major problem of mathematical platonism is this: precisely where and how do the mathematical entities exist, and how do we know about them? Is there a world, completely separate from our physical one that is occupied by the mathematical entities? How can we gain access to this separate world and discover truths about the entities? One answer might be Ultimate ensemble, which is a theory that postulates all structures that exist mathematically also exist physically in their own universe.

Plato spoke of mathematics by:

How do you mean?

I mean, as I was saying, that arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument. You know how steadily the masters of the art repel and ridicule any one who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue one and not become lost in fractions.

That is very true.

Now, suppose a person were to say to them: O my friends, what are these wonderful numbers about which you are reasoning, in which, as you say, there is a unity such as you demand, and each unit is equal, invariable, indivisible, --what would they answer?

—Plato, Chapter 7. "The Republic" (Jowell translation).

In context, chapter 8, H.D.P. Lee translation, reports the education of a philosopher containing five mathematical disciplines:

1. arithmetic, written in unit fraction 'parts' using theoretical unities and abstract numbers.
2. plane geometry and solid geometry also considered the line to be segmented into rational and irrational unit 'parts',
3. astronomy
4. harmonics

Translators of the works of Plato rebelled against practical versions of his culture's practical mathematics. However, Plato himself and Greeks had copied 1,500 older Egyptian fraction abstract unities, one being a hekat unity scaled to (64/64) in the Akhmim Wooden Tablet, thereby not getting lost in fractions.

Gödel's platonism postulates a special kind of mathematical intuition that lets us perceive mathematical objects directly. (This view bears resemblances to many things Husserl said about mathematics, and supports Kant's idea that mathematics is synthetic a priori.) Davis and Hersh have suggested in their book The Mathematical Experience that most mathematicians act as though they are Platonists, even though, if pressed to defend the position carefully, they may retreat to formalism (see below).

Some mathematicians hold opinions that amount to more nuanced versions of Platonism.

Full-blooded Platonism is a modern variation of Platonism, which is in reaction to the fact that different sets of mathematical entities can be proven to exist depending on the axioms and inference rules employed (for instance, the law of the excluded middle, and the axiom of choice). It holds that all mathematical entities exist, however they may be provable, even if they cannot all be derived from a single consistent set of axioms.

Empiricism

Empiricism is a form of realism that denies that mathematics can be known a priori at all. It says that we discover mathematical facts by empirical research, just like facts in any of the other sciences. It is not one of the classical three positions advocated in the early 20th century, but primarily arose in the middle of the century. However, an important early proponent of a view like this was John Stuart Mill. Mill's view was widely criticized, because it makes statements like "2 + 2 = 4" come out as uncertain, contingent truths, which we can only learn by observing instances of two pairs coming together and forming a quartet.

Contemporary mathematical empiricism, formulated by Quine and Putnam, is primarily supported by the indispensability argument: mathematics is indispensable to all empirical sciences, and if we want to believe in the reality of the phenomena described by the sciences, we ought also believe in the reality of those entities required for this description. That is, since physics needs to talk about electrons to say why light bulbs behave as they do, then electrons must exist. Since physics needs to talk about numbers in offering any of its explanations, then numbers must exist. In keeping with Quine and Putnam's overall philosophies, this is a naturalistic argument. It argues for the existence of mathematical entities as the best explanation for experience, thus stripping mathematics of some of its distinctness from the other sciences.

Putnam strongly rejected the term "Platonist" as implying an overly-specific ontology that was not necessary to mathematical practice in any real sense. He advocated a form of "pure realism" that rejected mystical notions of truth and accepted much quasi-empiricism in mathematics. Putnam was involved in coining the term "pure realism" (see below).

The most important criticism of empirical views of mathematics is approximately the same as that raised against Mill. If mathematics is just as empirical as the other sciences, then this suggests that its results are just as fallible as theirs, and just as contingent. In Mill's case the empirical justification comes directly, while in Quine's case it comes indirectly, through the coherence of our scientific theory as a whole, i.e. consilience after E O Wilson. Quine suggests that mathematics seems completely certain because the role it plays in our web of belief is incredibly central, and that it would be extremely difficult for us to revise it, though not impossible.

For a philosophy of mathematics that attempts to overcome some of the shortcomings of Quine and Gödel's approaches by taking aspects of each see Penelope Maddy's Realism in Mathematics. Another example of a realist theory is the embodied mind theory (below). For a modern revision of mathematical empiricism see New Empiricism (below).

For experimental evidence suggesting that one-day-old babies can do elementary arithmetic, see Brian Butterworth.

Mathematical Monism

Max Tegmark's Mathematical universe hypothesis goes further than full-blooded Platonism in asserting that not only do all mathematical objects exist, but nothing else does. Tegmark's sole postulate is: All structures that exist mathematically also exist physically. That is, in the sense that "in those [worlds] complex enough to contain self-aware substructures [they] will subjectively perceive themselves as existing in a physically 'real' world".[3][4]

Logicism

Logicism is the thesis that mathematics is reducible to logic, and hence nothing but a part of logic (Carnap 1931/1883, 41). Logicists hold that mathematics can be known a priori, but suggest that our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. In this view, logic is the proper foundation of mathematics, and all mathematical statements are necessary logical truths.

Rudolf Carnap (1931) presents the logicist thesis in two parts:

1. The concepts of mathematics can be derived from logical concepts through explicit definitions.
2. The theorems of mathematics can be derived from logical axioms through purely logical deduction.

Gottlob Frege was the founder of logicism. In his seminal Die Grundgesetze der Arithmetik (Basic Laws of Arithmetic) he built up arithmetic from a system of logic with a general principle of comprehension, which he called "Basic Law V" (for concepts F and G, the extension of F equals the extension of G if and only if for all objects a, Fa if and only if Ga), a principle that he took to be acceptable as part of logic.

Frege's construction was flawed. Russell discovered that Basic Law V is inconsistent. (This is Russell's paradox) Frege abandoned his logicist program soon after this, but it was continued by Russell and . They attributed the paradox to "vicious circularity" and built up what they called ramified type theory to deal with it. In this system, they were eventually able to build up much of modern mathematics but in an altered, and excessively complex, form (for example, there were different natural numbers in each type, and there were infinitely many types). They also had to make several compromises in order to develop so much of mathematics, such as an "axiom of reducibility". Even Russell said that this axiom did not really belong to logic.

Modern logicists (like Bob Hale, Crispin Wright, and perhaps others) have returned to a program closer to Frege's. They have abandoned Basic Law V in favour of abstraction principles such as Hume's principle (the number of objects falling under the concept F equals the number of objects falling under the concept G if and only if the extension of F and the extension of G can be put into one-to-one correspondence). Frege required Basic Law V to be able to give an explicit definition of the numbers, but all the properties of numbers can be derived from Hume's principle. This would not have been enough for Frege because (to paraphrase him) it does not exclude the possibility that the number 3 is in fact Julius Caesar. In addition, many of the weakened principles that they have had to adopt to replace Basic Law V no longer seem so obviously analytic, and thus purely logical.

If mathematics is a part of logic, then questions about mathematical objects reduce to questions about logical objects. But what, one might ask, are the objects of logical concepts? In this sense, logicism can be seen as shifting questions about the philosophy of mathematics to questions about logic without fully answering them.

Formalism

Formalism holds that mathematical statements may be thought of as statements about the consequences of certain string manipulation rules. For example, in the "game" of Euclidean geometry (which is seen as consisting of some strings called "axioms", and some "rules of inference" to generate new strings from given ones), one can prove that the Pythagorean theorem holds (that is, you can generate the string corresponding to the Pythagorean theorem). According to Formalism, mathematical truths are not about numbers and sets and triangles and the like — in fact, they aren't "about" anything at all.

Another version of formalism is often known as deductivism. In deductivism, the Pythagorean theorem is not an absolute truth, but a relative one: if you assign meaning to the strings in such a way that the rules of the game become true (i.e., true statements are assigned to the axioms and the rules of inference are truth-preserving), then you have to accept the theorem, or, rather, the interpretation you have given it must be a true statement. The same is held to be true for all other mathematical statements. Thus, formalism need not mean that mathematics is nothing more than a meaningless symbolic game. It is usually hoped that there exists some interpretation in which the rules of the game hold. (Compare this position to structuralism.) But it does allow the working mathematician to continue in his or her work and leave such problems to the philosopher or scientist. Many formalists would say that in practice, the axiom systems to be studied will be suggested by the demands of science or other areas of mathematics.

A major early proponent of formalism was David Hilbert, whose program was intended to be a complete and consistent axiomatization of all of mathematics. ("Consistent" here means that no contradictions can be derived from the system.) Hilbert aimed to show the consistency of mathematical systems from the assumption that the "finitary arithmetic" (a subsystem of the usual arithmetic of the positive integers, chosen to be philosophically uncontroversial) was consistent.

Hilbert's goals of creating a system of mathematics that is both complete and consistent were dealt a fatal blow by the second of Gödel's incompleteness theorems, which states that sufficiently expressive consistent axiom systems can never prove their own consistency. Since any such axiom system would contain the finitary arithmetic as a subsystem, Gödel's theorem implied that it would be impossible to prove the system's consistency relative to that (since it would then prove its own consistency, which Gödel had shown was impossible). Thus, in order to show that any axiomatic system of mathematics is in fact consistent, one needs to first assume the consistency of a system of mathematics that is in a sense stronger than the system to be proven consistent.

Hilbert was initially a deductivist, but, as may be clear from above, he considered certain metamathematical methods to yield intrinsically meaningful results and was a realist with respect to the finitary arithmetic. Later, he held the opinion that there was no other meaningful mathematics whatsoever, regardless of interpretation.

Other formalists, such as Rudolf Carnap, Alfred Tarski and Haskell Curry, considered mathematics to be the investigation of formal axiom systems. Mathematical logicians study formal systems but are just as often realists as they are formalists.

Formalists are relatively tolerant and inviting to new approaches to logic, non-standard number systems, new set theories etc. The more games we study, the better. However, in all three of these examples, motivation is drawn from existing mathematical or philosophical concerns. The "games" are usually not arbitrary.

The main critique of formalism is that the actual mathematical ideas that occupy mathematicians are far removed from the string manipulation games mentioned above. Formalism is thus silent on the question of which axiom systems ought to be studied, as none is more meaningful than another from a formalistic point of view.

Recently, some formalist mathematicians have proposed that all of our formal mathematical knowledge should be systematically encoded in computer-readable formats, so as to facilitate automated proof checking of mathematical proofs and the use of interactive theorem proving in the development of mathematical theories and computer software. Because of their close connection with computer science, this idea is also advocated by mathematical intuitionists and constructivists in the "computability" tradition (see below). See QED project for a general overview.

Conventionalism

The French mathematician Henri Poincaré was among the first to articulate a conventionalist view. Poincaré's use of non-Euclidean geometries in his work on differential equations convinced him that Euclidean geometry should not be regarded as a priori truth. He held that axioms in geometry should be chosen for the results they produce, not for their apparent coherence with human intuitions about the physical world.

Psychologism

Psychologism in the philosophy of mathematics is the position that mathematical concepts and/or truths are grounded in, derived from or explained by psychological facts (or laws).

John Stuart Mill seems to have been an advocate of a type of logical psychologism, as were many nineteenth-century German logicians such as Sigwart and Erdmann as well as a number of psychologists, past and present: for example, Gustave Le Bon. Psychologism was famously criticized by Frege in his The Foundations of Arithmetic, and many of his works and essays, including his review of Philosophy of Arithmetic. Edmund Husserl, in the first volume of his Logical Investigations, called "The Prolegomena of Pure Logic", criticized psychologism thoroughly and sought to distance himself from it. The "Prolegomena" is considered a more concise, fair, and thorough refutation of psychologism than the criticisms made by Frege, and also it is considered today by many as being a memorable refutation for its decisive blow to psychologism. Psychologism was also criticized by Charles Sanders Peirce and Maurice Merleau-Ponty.

Intuitionism

In mathematics, intuitionism is a program of methodological reform whose motto is that "there are no non-experienced mathematical truths" (L.E.J. Brouwer). From this springboard, intuitionists seek to reconstruct what they consider to be the corrigible portion of mathematics in accordance with Kantian concepts of being, becoming, intuition, and knowledge. Brouwer, the founder of the movement, held that mathematical objects arise from the a priori forms of the volitions that inform the perception of empirical objects. (CDP, 542)

A major force behind Intuitionism was L.E.J. Brouwer, who rejected the usefulness of formalized logic of any sort for mathematics. His student Arend Heyting postulated an intuitionistic logic, different from the classical Aristotelian logic; this logic does not contain the law of the excluded middle and therefore frowns upon proofs by contradiction. The axiom of choice is also rejected in most intuitionistic set theories, though in some versions it is accepted. Important work was later done by Errett Bishop, who managed to prove versions of the most important theorems in real analysis within this framework.

In intuitionism, the term "explicit construction" is not cleanly defined, and that has led to criticisms. Attempts have been made to use the concepts of Turing machine or computable function to fill this gap, leading to the claim that only questions regarding the behavior of finite algorithms are meaningful and should be investigated in mathematics. This has led to the study of the computable numbers, first introduced by Alan Turing. Not surprisingly, then, this approach to mathematics is sometimes associated with theoretical computer science

Constructivism

Like intuitionism, constructivism involves the regulative principle that only mathematical entities which can be explicitly constructed in a certain sense should be admitted to mathematical discourse. In this view, mathematics is an exercise of the human intuition, not a game played with meaningless symbols. Instead, it is about entities that we can create directly through mental activity. In addition, some adherents of these schools reject non-constructive proofs, such as a proof by contradiction.

Finitism

Finitism is an extreme form of constructivism, according to which a mathematical object does not exist unless it can be constructed from natural numbers in a finite number of steps. In her book Philosophy of Set Theory, Mary Tiles characterized those who allow countably infinite objects as classical finitists, and those who deny even countably infinite objects as strict finitists.

The most famous proponent of finitism was Leopold Kronecker,[5] who said:

God created the natural numbers, all else is the work of man.

Ultrafinitism is an even more extreme version of finitism, which rejects not only infinities but finite quantities that cannot feasibly be constructed with available resources.

Structuralism

Structuralism is a position holding that mathematical theories describe structures, and that mathematical objects are exhaustively defined by their places in such structures, consequently having no intrinsic properties. For instance, it would maintain that all that needs to be known about the number 1 is that is it's the first whole number after 0. Likewise all the other whole numbers are defined by their places in a structure, the number line. Other examples of mathematical objects might include lines and planes in geometry, or elements and operations in abstract algebra.

Structuralism is a epistemologically realistic view in that it holds that mathematical statements have an objective truth value. However, its central claim only relates to what kind of entity a mathematical object is, not to what kind of existence mathematical objects or structures have (not, in other words, to their ontology). The kind of existence mathematical objects have would clearly be dependent on that of the structures in which they are embedded; different sub-varieties of structuralism make different ontological claims in this regard.[6]

The Ante Rem, or fully realist, variation of structuralism has a similar ontology to Platonism in that structures are held to have a real but abstract and immaterial existence. As such, it faces the usual problems of explaining the interaction between such abstract structures and flesh-and-blood mathematicians.

In Re, or moderately realistic, structuralism is the equivalent of Aristotelean realism. Structures are held to exist inasmuch as some concrete system exemplifies them. This incurs the usual issues that some perfectly legitimate structures might accidentally happen not to exist, and that a finite physical world might not be "big" enough to accommodate some otherwise legitimate structures.

The Post Res or eliminative variant of structuralism is anti-realist about structures in a way that parallels nominalism. According to this view mathematical systems exist, and have structural features in common. If something is true of a structure, it will be true of all systems exemplifying the structure. However, it is merely convenient to talk of structures being "held in common" between systems: they in fact have no independent existence.

Embodied mind theories

Embodied mind theories hold that mathematical thought is a natural outgrowth of the human cognitive apparatus which finds itself in our physical universe. For example, the abstract concept of number springs from the experience of

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counting discrete objects. It is held that mathematics is not universal and does not exist in any real sense, other than in human brains. Humans construct, but do not discover, mathematics.

With this view, the physical universe can thus be seen as the ultimate foundation of mathematics: it guided the evolution of the brain and later determined which questions this brain would find worthy of investigation. However, the human mind has no special claim on reality or approaches to it built out of math. If such constructs as Euler's identity are true then they are true as a map of the human mind and cognition.

Embodied mind theorists thus explain the effectiveness of mathematics — mathematics was constructed by the brain in order to be effective in this universe.

The most accessible, famous, and infamous treatment of this perspective is Where Mathematics Comes From, by and Rafael E. Núñez. In addition, mathematician Keith Devlin has investigated similar concepts with his book The Math Instinct. For more on the philosophical ideas that inspired this perspective, see cognitive science of mathematics.

New Empiricism

A more recent empiricism returns to the principle of the English empiricists of the 18th and 19th Centuries, in particular John Stuart Mill, who asserted that all knowledge comes to us from observation through the senses. This applies not only to matters of fact, but also to "relations of ideas," as Hume called them: the structures of logic which interpret, organize and abstract observations.

To this principle it adds a materialist connection: All the processes of logic which interpret, organize and abstract observations, are physical phenomena which take place in real time and physical space: namely, in the brains of human beings. Abstract objects, such as mathematical objects, are ideas, which in turn exist as electrical and chemical states of the billions of neurons in the human brain.

This second concept is reminiscent of the social constructivist approach, which holds that mathematics is produced by humans rather than being “discovered” from abstract, a priori truths. However, it differs sharply from the constructivist implication that humans arbitrarily construct mathematical principles that have no inherent truth but which instead are created on a conveniency basis. On the contrary, new empiricism shows how mathematics, although constructed by humans, follows rules and principles that will be agreed on by all who participate in the process, with the result that everyone practicing mathematics comes up with the same answer — except in those areas where there is philosophical disagreement on the meaning of fundamental concepts. This is because the new empiricism perceives this agreement as being a physical phenomenon. One which is observed by other humans in the same way that other physical phenomena, like the motions of inanimate bodies, or the chemical interaction of various elements, are observed.

A difficulty lies in the observation that mathematical truths based on logical deduction appear to be more certainly true than knowledge of the physical world itself. (The physical world in this case is taken to mean the portion of it lying outside the human brain.)

Kant argued that the structures of logic which organize, interpret and abstract observations were built into the human mind and were true and valid a priori. Mill, on the contrary, said that we believe them to be true because we have enough individual instances of their truth to generalize: in his words, "From instances we have observed, we feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present and future, however numerous they may be."[7] Although the psychological or epistemological specifics given by Mill through which we build our logical apparatus may not be completely warranted, his explanation still nonetheless manages to demonstrate that there is no way around Kant's a priori logic. To recant Mill's original idea in an empiricist twist: “Indeed, the very principles of logical deduction are true because we observe that using them leads to true conclusions.”, which is itself an a priori pressupposition.

For most mathematicians the empiricist principle that all knowledge comes from the senses contradicts a more basic principle: that mathematical propositions are true independent of the physical world. Everything about a mathematical proposition is independent of what appears to be the physical world. It all takes place in the mind. And the mind operates on infallible principles of deductive logic. It is not influenced by exterior inputs from the physical world, distorted by having to pass through the tentative, contingent universe of the senses. It all happens internally, so to say. This in turn may be the answer to what brings about Gödel's special kind of mathematical intuition, which was mentioned earlier in the article.

If all this is true, then where do the world senses come in? The early empiricists all stumbled over this point. Hume asserted that all knowledge comes from the senses, and then gave away the ballgame by excepting abstract propositions, which he called "relations of ideas." These, he said, were absolutely true (although the mathematicians who thought them up, being human, might get them wrong). Mill, on the other hand, tried to deny that abstract ideas exist outside the physical world: all numbers, he said, "must be numbers of something: there are no such things as numbers in the abstract." When we count to eight or add five and three we are really counting spoons or bumblebees. "All things possess quantity," he said, so that propositions concerning numbers are propositions concerning "all things whatever." But then in almost a contradiction of himself he went on to acknowledge that numerical and algebraic expressions are not necessarily attached to real world objects: they "do not excite in our minds ideas of any things in particular." Mill's low reputation as a philosopher of logic, and the low estate of empiricism in the century and a half following him, derives from this failed attempt to link abstract thoughts to the physical world, when it is obvious that abstraction consists precisely of separating the thought from its physical foundations.

The conundrum created by our certainty that abstract deductive propositions, if valid (i.e., if we can "prove" them), are true, exclusive of observation and testing in the physical world, gives rise to a further reflection...What if thoughts themselves, and the minds that create them, are physical objects, existing only in the physical world?

This would not reconcile the contradiction between our belief in the certainty of abstract deductions and the empiricist principle that knowledge comes from observation of individual instances. We know that Euler's equation is true because every time a human mind derives the equation, it gets the same result, unless it has made a mistake, which can be acknowledged and corrected. We observe this phenomenon, and we extrapolate to the general proposition that it is always true. However, based on this rationale, one would still not be warranted in concluding that mathematics are purely empirical in nature.

This applies not only to physical principles, like the law of gravity, but to abstract phenomena that

Aristotelian realism

Similar to empiricism in emphasizing the relation of mathematics to the real world, Aristotelian realism holds that mathematics studies properties such as symmetry, continuity and order that can be literally realized in the physical world (or in any other world there might be). It contrasts with Platonism in holding that the objects of mathematics, such as numbers, do not exist in an "abstract" world but can be physically realized. For example, the number 4 is realized in the relation between a heap of parrots and the universal "being a parrot" that divides the heap into so many parrots.[8] Aristotelian realism is defended by James Franklin and the Sydney School in the philosophy of mathematics and is close to the view of Penelope Maddy (1990) that when I open an egg carton I perceive a set of three eggs (that is, a mathematical entity realized in the physical world). A problem for Aristotelian realism is what account to give of higher infinities, which may not be realizable in the physical world.

Fictionalism

Fictionalism in mathematics was brought to fame in 1980 when Hartry Field published Science Without Numbers, which rejected and in fact reversed Quine's indispensability argument. Where Quine suggested that mathematics was indispensable for our best scientific theories, and therefore should be accepted as a body of truths talking about independently existing entities, Field suggested that mathematics was dispensable, and therefore should be considered as a body of falsehoods not talking about anything real. He did this by giving a complete axiomatization of Newtonian mechanics that didn't reference numbers or functions at all. He started with the "betweenness" of Hilbert's axioms to characterize space without coordinatizing it, and then added extra relations between points to do the work formerly done by vector fields. Hilbert's geometry is mathematical, because it talks about abstract points, but in Field's theory, these points are the concrete points of physical space, so no special mathematical objects at all are needed.

Having shown how to do science without using mathematics, Field proceeded to rehabilitate mathematics as a kind of useful fiction. He showed that mathematical physics is a conservative extension of his non-mathematical physics (that is, every physical fact provable in mathematical physics is already provable from Field's system), so that the mathematics is a reliable process whose physical applications are all true, even though its own statements are false. Thus, when doing mathematics, we can see ourselves as telling a sort of story, talking as if numbers existed. For Field, a statement like "2 + 2 = 4" is just as fictitious as "Sherlock Holmes lived at 221B Baker Street" — but both are true according to the relevant fictions. By this account, there are no metaphysical or epistemological problems special to mathematics. The only worries left are the general worries about non-mathematical physics, and about fiction in general. Field's approach has been very influential, but is widely rejected. This is in part because of the requirement of strong fragments of second-order logic to carry out his reduction, and because the statement of conservativity seems to require quantification over abstract models or deductions.

Social constructivism or social realism

Social constructivism or social realism theories see mathematics primarily as a social construct, as a product of culture, subject to correction and change. Like the other sciences, mathematics is viewed as an empirical endeavor whose results are constantly evaluated and may be discarded. However, while on an empiricist view the evaluation is some sort of comparison with "reality", social constructivists emphasize that the direction of mathematical research is dictated by the fashions of the social group performing it or by the needs of the society financing it. However, although such external forces may change the direction of some mathematical research, there are strong internal constraints — the mathematical traditions, methods, problems, meanings and values into which mathematicians are enculturated — that work to conserve the historically defined discipline.

This runs counter to the traditional beliefs of working mathematicians, that mathematics is somehow pure or objective. But social constructivists argue that mathematics is in fact grounded by much uncertainty: as mathematical practice evolves, the status of previous mathematics is cast into doubt, and is corrected to the degree it is required or desired by the current mathematical community. This can be seen in the development of analysis from reexamination of the calculus of Leibniz and Newton. They argue further that finished mathematics is often accorded too much status, and folk mathematics not enough, due to an over-emphasis on axiomatic proof and peer review as practices. However, this might be seen as merely saying that rigorously proven results are overemphasized, and then "look how chaotic and uncertain the rest of it all is!"

The social nature of mathematics is highlighted in its subcultures. Major discoveries can be made in one branch of mathematics and be relevant to another, yet the relationship goes undiscovered for lack of social contact between mathematicians. Social constructivists argue each speciality forms its own epistemic community and often has great difficulty communicating, or motivating the investigation of unifying conjectures that might relate different areas of mathematics. Social constructivists see the process of "doing mathematics" as actually creating the meaning, while social realists see a deficiency either of human capacity to abstractify, or of human's cognitive bias, or of mathematicians' collective intelligence as preventing the comprehension of a real universe of mathematical objects. Social constructivists sometimes reject the search for foundations of mathematics as bound to fail, as pointless or even meaningless. Some social scientists also argue that mathematics is not real or objective at all, but is affected by racism and ethnocentrism. Some of these ideas are close to postmodernism.

Contributions to this school have been made by Imre Lakatos and Thomas Tymoczko, although it is not clear that either would endorse the title. More recently Paul Ernest has explicitly formulated a social constructivist philosophy of mathematics.[9] Some consider the work of Paul Erdős as a whole to have advanced this view (although he personally rejected it) because of his uniquely broad collaborations, which prompted others to see and study "mathematics as a social activity", e.g., via the Erdős number. Reuben Hersh has also promoted the social view of mathematics, calling it a "humanistic" approach,[10] similar to but not quite the same as that associated with Alvin White;[11] one of Hersh's co-authors, Philip J. Davis, has expressed sympathy for the social view as well.

A criticism of this approach is that it is trivial, based on the trivial observation that mathematics is a human activity. To observe that rigorous proof comes only after unrigorous conjecture, experimentation and speculation is true, but it is trivial and no-one would deny this. So it's a bit of a stretch to characterize a philosophy of mathematics in this way, on something trivially true. The calculus of Leibniz and Newton was reexamined by mathematicians such as Weierstrass in order to rigorously prove the theorems thereof. There is nothing special or interesting about this, as it fits in with the more

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general trend of unrigorous ideas which are later made rigorous. There needs to be a clear distinction between the objects of study of mathematics and the study of the objects of study of mathematics. The former doesn't seem to change a great deal the latter is forever in flux. The latter is what the Social theory is about, and the former is what Platonism et al. are about.

However, this criticism is rejected by supporters of the social constructivist perspective because it misses the point that the very objects of mathematics are social constructs. These objects, it asserts, are primarily semiotic objects existing in the sphere of human culture, sustained by social practices (after Wittgenstein) that utilize physically embodied signs and give rise to intrapersonal (mental) constructs. Social constructivists view the reification of the sphere of human culture into a Platonic realm, or some other heaven-like domain of existence beyond the physical world, a long standing category error.

Beyond the traditional schools

Rather than focus on narrow debates about the true nature of mathematical truth, or even on practices unique to mathematicians such as the proof, a growing movement from the 1960s to the 1990s began to question the idea of seeking foundations or finding any one right answer to why mathematics works. The starting point for this was Eugene Wigner's famous 1960 paper The Unreasonable Effectiveness of Mathematics in the Natural Sciences, in which he argued that the happy coincidence of mathematics and physics being so well matched seemed to be unreasonable and hard to explain.

The embodied-mind or cognitive school and the social school were responses to this challenge, but the debates raised were difficult to confine to those.

Quasi-empiricism

One parallel concern that does not actually challenge the schools directly but instead questions their focus is the notion of quasi-empiricism in mathematics. This grew from the increasingly popular assertion in the late 20th century that no one foundation of mathematics could be ever proven to exist. It is also sometimes called "postmodernism in mathematics" although that term is considered overloaded by some and insulting by others. Quasi-empiricism argues that in doing their research, mathematicians test hypotheses as well as prove theorems. A mathematical argument can transmit falsity from the conclusion to the premises just as well as it can transmit truth from the premises to the conclusion. Quasi-empiricism was developed by Imre Lakatos, inspired by the philosophy of science of Karl Popper.

Lakatos' philosophy of mathematics is sometimes regarded as a kind of social constructivism, but this was not his intention.

Such methods have always been part of folk mathematics by which great feats of calculation and measurement are sometimes achieved. Indeed, such methods may be the only notion of proof a culture has.

Hilary Putnam has argued that any theory of mathematical realism would include quasi-empirical methods. He proposed that an alien species doing mathematics might well rely on quasi-empirical methods primarily, being willing often to forgo rigorous and axiomatic proofs, and still be doing mathematics — at perhaps a somewhat greater risk of failure of their calculations. He gave a detailed argument for this in New Directions (ed. Tymockzo, 1998).

Popper's "two senses" theory

Realist and constructivist theories are normally taken to be contraries. However, Karl Popper[12] argued that a number statement such as "2 apples + 2 apples = 4 apples" can be taken in two senses. In one sense it is irrefutable and logically true in the second sense it is factually true and falsifiable. Another way of putting this is to say that a single number statement can express two proposition one of which can be explained on constructivist lines the other on realist lines.[13]

Unification

Few philosophers are able to penetrate mathematical notations and culture to relate conventional notions of metaphysics to the more specialized metaphysical notions of the schools above. This may lead to a disconnection in which some mathematicians continue to profess discredited philosophy as a justification for their continued belief in a world-view promoting their work.

Although the social theories and quasi-empiricism, and especially the embodied mind theory, have focused more attention on the epistemology implied by current mathematical practices, they fall far short of actually relating this to ordinary human perception and everyday understandings of knowledge.

Language

Innovations in the philosophy of language during the 20th century renewed interest in whether mathematics is, as is often said, the language of science. Although most mathematicians and physicists (and many philosophers) would accept the statement "mathematics is a language", linguists believe that the implications of such a statement must be considered. For example, the tools of linguistics are not generally applied to the symbol systems of mathematics, that is, mathematics is studied in a markedly different way than other languages. If mathematics is a language, it is a different type of language than natural languages. Indeed, because of the need for clarity and specificity, the language of mathematics is far more constrained than natural languages studied by linguists. However, the methods developed by Frege and Tarski for the study of mathematical language have been extended greatly by Tarski's student Richard Montague and other linguists working in formal semantics to show that the distinction between mathematical language and natural language may not be as great as it seems.

Arguments

This argument, associated with Willard Quine and Hilary Putnam, is considered by Stephen Yablo to be one of the most challenging arguments in favor of the acceptance of the existence of abstract mathematical entities, such as numbers and sets.[14] The form of the argument is as follows.

1. One must have commitments to all entities that are indispensable to the best scientific theories, and to those entities only (commonly referred to as "all and only").
2. Mathematical entities are indispensable to the best scientific theories. Therefore,
3. One must have ontological commitments to mathematical entities.[15]

The justification for the first premise is the most controversial. Both Putnam and Quine invoke naturalism to justify the exclusion of all non-scientific entities, and hence to defend the "only" part of "all and only". The assertion that "all" entities postulated in scientific theories, including numbers, should be accepted as real is justified by confirmation holism. Since theories are not confirmed in a piecemeal fashion, but as a whole, there is no justification for excluding any of the entities referred to in well-confirmed theories. This puts the nominalist who wishes to exclude the existence of sets and non-Euclidean geometry, but to include the existence of quarks and other undetectable entities of physics, for example, in a difficult position.

The epistemic argument against realism

The anti-realist "epistemic argument" against Platonism has been made by Paul Benacerraf and Hartry Field. Platonism posits that mathematical objects are abstract entities. By general agreement, abstract entities cannot interact causally with concrete, physical entities. ("the truth-values of our mathematical assertions depend on facts involving platonic entities that reside in a realm outside of space-time"[16]) Whilst our knowledge of concrete, physical objects is based on our ability to perceive them, and therefore to causally interact with them, there is no parallel account of how mathematicians come to have knowledge of abstract objects.[17][18][19] ("An account of mathematical truth ..must be consistent with the possibility of mathematical knowledge"[20]). Another way of making the point is that if the Platonic world were to disappear, it would make no difference to the ability of mathematicians to generate proofs, etc., which is already fully accountable in terms of physical processes in their brains.

Field developed his views into fictionalism. Benacerraf also developed the philosophy of mathematical structuralism, according to which there are no mathematical objects. Nonetheless, some versions of structuralism are compatible with some versions of realism.

The argument hinges on the idea that a satisfactory naturalistic account of thought processes in terms of brain processes can be given for mathematical reasoning along with everything else. One line of defence is to maintain that this is false, so that mathematical reasoning uses some special intuition that involves contact with the Platonic realm. A modern form of this argument is given by .[21]

Another line of defence is to maintain that abstract objects are relevant to mathematical reasoning in a way that is non causal, and not analogous to perception. This argument is developed by Jerrold Katz in his book Realistic Rationalism.

A more radical defense is denial of physical reality, i.e. the mathematical universe hypothesis. In that case, a mathematicians knowledge of mathematics is one mathematical object making contact with another.

Aesthetics

Many practising mathematicians have been drawn to their subject because of a sense of beauty they perceive in it. One sometimes hears the sentiment that mathematicians would like to leave philosophy to the philosophers and get back to mathematics — where, presumably, the beauty lies.

In his work on the divine proportion, H. E. Huntley relates the feeling of reading and understanding someone else's proof of a theorem of mathematics to that of a viewer of a masterpiece of art — the reader of a proof has a similar sense of exhilaration at understanding as the original author of the proof, much as, he argues, the viewer of a masterpiece has a sense of exhilaration similar to the original painter or sculptor. Indeed, one can study mathematical and scientific writings as literature.

Philip J. Davis and Reuben Hersh have commented that the sense of mathematical beauty is universal amongst practicing mathematicians. By way of example, they provide two proofs of the irrationality of the $\sqrt{2}$. The first is the traditional proof by , ascribed to Euclid; the second is a more direct proof involving the fundamental theorem of arithmetic that, they argue, gets to the heart of the issue. Davis and Hersh argue that mathematicians find the second proof more aesthetically appealing because it gets closer to the nature of the problem.

Paul Erdős was well-known for his notion of a hypothetical "Book" containing the most elegant or beautiful mathematical proofs. There is not universal agreement that a result has one "most elegant" proof; Gregory Chaitin has argued against this idea.

Philosophers have sometimes criticized mathematicians' sense of beauty or elegance as being, at best, vaguely stated. By the same token, however, philosophers of mathematics have sought to characterize what makes one proof more desirable than another when both are logically sound.

Another aspect of aesthetics concerning mathematics is mathematicians' views towards the possible uses of mathematics for purposes deemed unethical or inappropriate. The best-known exposition of this view occurs in G.H. Hardy's book A Mathematician's Apology, in which Hardy argues that pure mathematics is superior in beauty to applied mathematics precisely because it cannot be used for war and similar ends. Some later mathematicians have characterized Hardy's views as mildly dated with the applicability of number theory to modern-day .

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